SOCIAL NETWORKS

LECTURE # 12

0-1 LAWS: (Tipping Point / Phase Transⁿ)

- Describe a phenomenon,
  where an event occurs or
  does not occur almost surely (a.s.).
- With a small change in the value of
  a critical parameter, the event
  of interest occurs either
  ALMOST NEVER (prob = 0)
  or ALMOST SURELY (prob = 1)

  $\boxed{\text{0-1 LAW}}$

⇒ The transition in probability
occurring very quickly. (VIRAL)

GAME OF "FRIENDING"

- Imagine sending a friend request
  randomly to $(n-1)$ other individuals
  in a network (with a total of n
  individuals).

Key Assumptions:

(i) If the recipient is already a friend, he simply ignores the request.

(ii) Otherwise, he receives your request and accepts you as a friend.

(iii) Under no circumstances, does he ignore, decline or unfriend you.

How many requests should you make to befriend all the other $(n-1)$ individuals?

TIPPING POINT.

After $\Theta(n \ln n)$ requests, one will have a.s. befriended all other $(n-1)$ individuals

COUPON COLLECTOR'S PROBLEM

"Collect-all-Coupons-and-Win" Contest.

Problem Statement.

- There are n distinct coupons.
- Coupons can be collected with replacement.

$$X_1, X_2, \ldots, X_n \sim \text{Bernoulli}(1/n).$$

i.i.d. r.v's.

$(x_i = 1) \equiv$ Event that you obtain the i^{th} coupon.

- What is the probability that more than t sample trials are needed to collect all n coupons?

$\equiv$ How many coupons are expected to be drawn with replacement <u>before</u> each of the n coupons has been drawn at least once?

Example:

Let $n = 52$
Then $t = 225$ $\Big\}$ $t = \Theta(n \ln n)$.

$\Rightarrow$ If you draw a card randomly (with replacement) from a full deck, then after $t = 225$ draws, you would have seen every card at least once a.s. (almost surely).

t_i = Time to collect i^{th} coupon after collecting $(i-1)^{th}$ coupon.

t_i's are independent.

$$t = \sum_{i=1}^{n} t_i = \text{Time to collect all coupons.}$$

$$p_i = \Pr[\text{Collect a new coupon after } (i-1)^{th}] = (n-i+1)/n$$

$t_i \sim \text{Geometric}(1/p_i)$

$$\Rightarrow \Pr[t_i = k] = (1-p_i)^k p_i$$

$$E(t_i) = \frac{1}{p_i} = \frac{n}{n-i+1}$$

$$\mathrm{Var}(t_i) = \frac{1}{p_i}\left(\frac{1}{p_i} - 1\right) = \frac{1-p_i}{p_i^2} = \frac{(i-1)n}{(n-i+1)^2}.$$

$$\le \frac{n^2}{(n-i+1)^2}$$

$E[t] = E[\Sigma t_i]$ By linearity of Expectation.

$= \frac{n}{n} + \frac{n}{n-1} + \cdots + \frac{n}{1}$

$= n\left(\frac{1}{n} + \frac{1}{n-1} + \cdots + 1\right) = nH_n$

$= n\int_1^n \frac{dx}{x} + \gamma n + 1/2 + o(n)$

$= n \ln n + O(n)$. $\gamma = 0.577$ = Euler's Const.

$Var(t) = Var[\Sigma t_i]$ $\because t_i$'s are independent.

$\leq \frac{n^2}{n^2} + \frac{n^2}{(n-1)^2} + \cdots + \frac{n^2}{1} = \frac{\pi^2}{6} n^2$

$\sigma(t) = \frac{\pi n}{\sqrt{6}}$

$Pr\left(|t - nH_n| \geq c.n\right)$

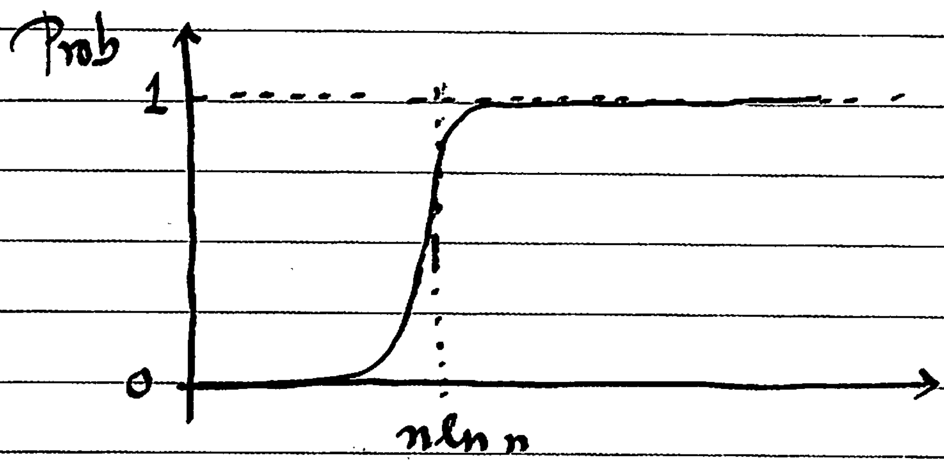
$= Pr\left(|t - nH_n| \geq \left(c\frac{\sqrt{6}}{\pi}\right)\sigma\right)$

$\leq \frac{\pi^2}{6c^2}$ (By Chebyshev's Inequality)

$Pr\left(|t - nH_n| \geq 10n\right) \leq \frac{\pi^2}{600} = \frac{1}{60}$.

$t < (1-\varepsilon)\, n H_n \Rightarrow$ You will not have all the coupons a.s.

$t > (1+\varepsilon)\, n H_n \Rightarrow$ You will have all the coupons a.s.



Consider a Random Graph $\sim G\left(n, \lambda \frac{\ln n}{n}\right)$.

$\lambda \approx 1 \longrightarrow$ Critical point (tipping).

<u>Indicator Variable</u>

$$\mathbb{1}_i = \begin{cases} 1 & \text{if node } i \text{ is isolated} \\ 0 & \text{otherwise.} \end{cases}$$

$\mathbb{1}_i \sim \text{Bernoulli}(\pi)$

$$\pi = \Pr[\mathbb{1}_i = 1] = \mathbb{E}[\mathbb{1}_i] = (1-p)^{n-1}$$

$$\approx e^{-np} = e^{-\lambda \ln n} = n^{-\lambda}.$$

$\therefore \ \mathbb{1}_i \sim \text{Bernoulli}(n^{-\lambda})$

$X = \sum \mathbb{1}_i$ = Total # isolated vertices.

$$E[X] = \sum E[\mathbb{1}_i] = n \cdot n^{-\lambda} = n^{1-\lambda}$$

$$Var[X] = \sum_i var[\mathbb{1}_i] + \sum_{i \neq j} Cov(\mathbb{1}_i \mathbb{1}_j)$$

$$= n \cdot n^{-\lambda} + n^2 n^{-2\lambda}\left(\lambda \frac{\ln n}{n}\right)$$

$$= n^{1-\lambda} + \lambda n^{1-2\lambda} \ln n.$$

If $\lambda > 1$ then $\mathbb{E}[X] = n^{1-\lambda} \to 0$.

$$\therefore Pr[X = 0] = 1.$$

$$(\because \mathbb{E}[X] = 0 \cdot Pr[X=0] + 1 \cdot Pr[X=1] \cdots)$$

If $\lambda < 1$ then $\mathbb{E}[X] = n^{1-\lambda}$, $Var[X] \approx n^{1-\lambda} \approx E[X]$

$$Pr[X = 0] = 0$$

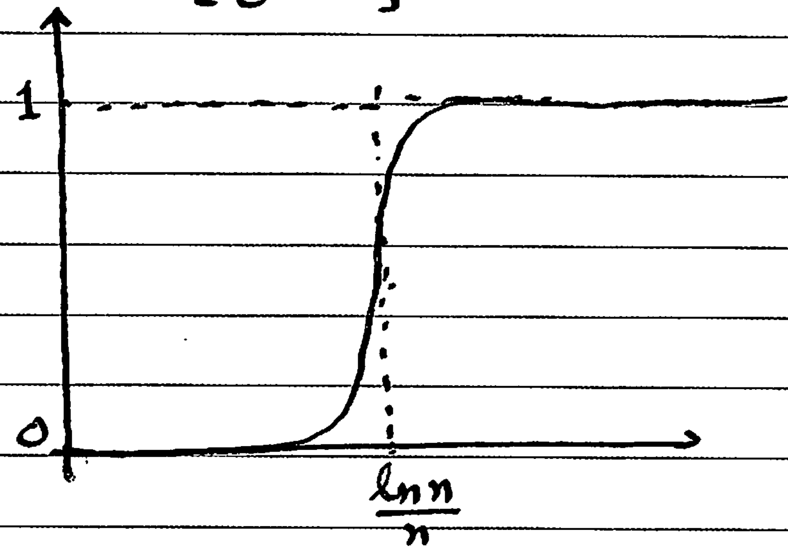
$$(\because Var[X] \geq (0 - E[X])^2 Pr[X=0] + \cdots$$

$$Var[X] \geq E[X]^2 Pr[X=0]$$

$$Pr[X=0] \leq \frac{Var[X]}{E[X]^2} = \frac{1}{E[X]}$$

$$= n^{-1+\lambda} \to 0.)$$

Pr[# Isolated Vertices $= 0$]



⇒ FURTHER ANALYSIS.

Pr[G = Connected]