Vectors and Dot Product

Basic Definitions

A *k-dimensional vector* is (for our purposes) a list of k numbers. We will use angle brackets to combine numbers into a vector; e.g. $\langle 3, 0, 1\rangle$ is a three-dimensional vector. Vectors are often notated by using a symbol with an arrow over it, such as $\vec{V}$.

Vectors of equal dimension can be added and subtracted and can be multiplied by numbers (called "scalars" in this context), by applying the operations component by component.
Let $\vec{U} = \langle u_1 \ldots u_k\rangle$ and $\vec{V} = \langle v_1 \ldots v_k\rangle$ be vectors and let p be a scalar.
Then $\vec{U} + \vec{V} = \langle u_1 + v_1 \ldots u_k + v_k\rangle$ and $p \cdot \vec{U} = \langle pu_1 \ldots pu_k\rangle$.

For example:

$$\langle 3, 0, 1\rangle + \langle 2, -1, 2\rangle = \langle 5, -1, 3\rangle.$$
$$\langle 3, 0, 1\rangle - \langle 2, -1, 2\rangle = \langle 1, 1, -1\rangle.$$
$$2 \cdot \langle 3, 0, 1\rangle = \langle 6, 0, 2\rangle.$$

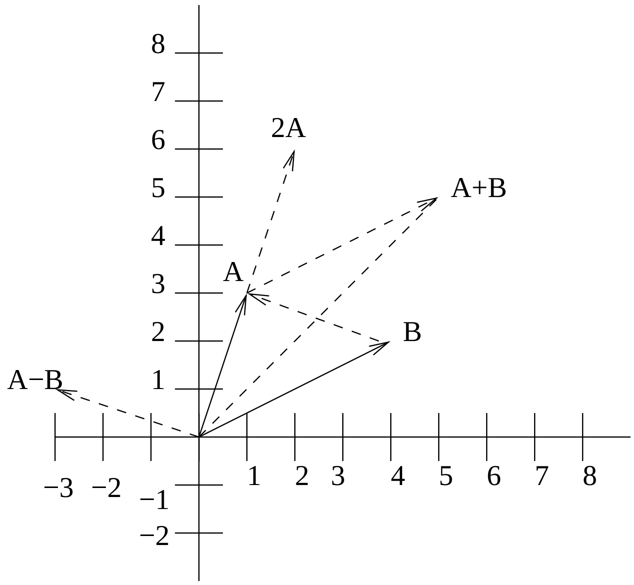
The *zero vector*, denoted $\vec{0}$, is the vector all of whose components are 0. (Strictly speaking, of course, there is a one-dimensional zero-vector, a two-dimensional zero vector, and so on, all of which are different. Which dimensionality is intended by the notation $\vec{0}$ is generally determined by context.)

Whenever two vectors are mentioned together, it is usually implicit that they have the same dimension; if they have different dimensions, this should be stated explicitly.

Geometric Interpretation

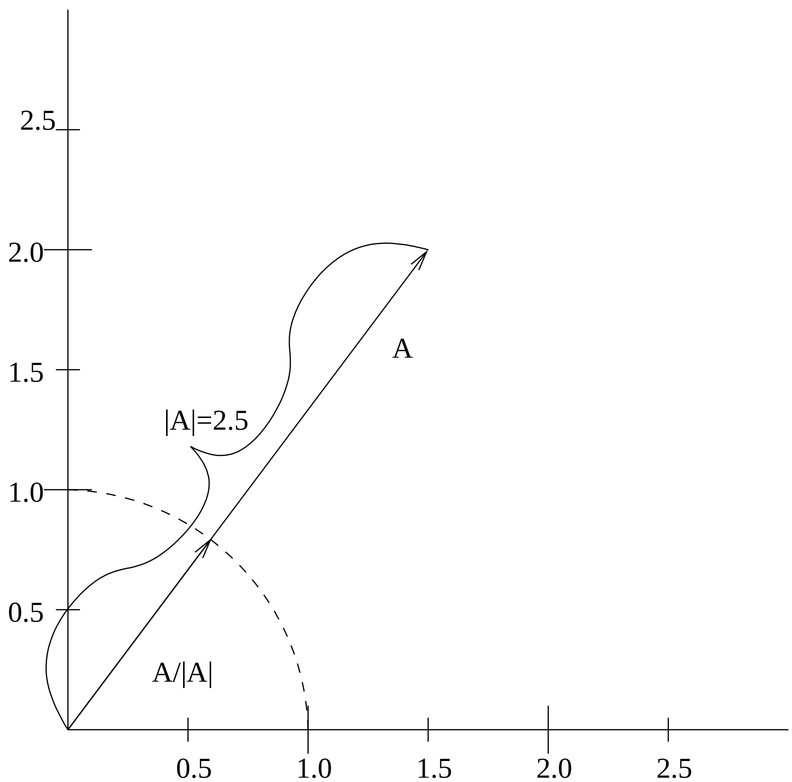
If you consider k-dimensional geometric space (usually $k = 2$ or 3, but sometimes more), and you fix a coordinate system, then a k-dimensional vector is an arrow from the origin to a point in space whose coordinates correspond to the elements of the vector. Therefore, the elements of a vector are often called its "coordinates".

Under this interpretation, the product $p \cdot \vec{V}$ is a vector aligned with V but p times as long. If $\vec{V} \neq \vec{0}$ then $\vec{V}$ and $p \cdot \vec{V}$ are said to be "parallel" if $p > 0$ and "anti-parallel" if $p < 0$. The sum $\vec{U} + \vec{V}$ corresponds to the following geometric construction: Draw an arrow parallel to $\vec{V}$ and the same length whose tail lies on the head of of $\vec{U}$. Then the head of this new arrow is at $\vec{U} + \vec{V}$.



$\vec{A} = \langle 1, 3 \rangle$. $\vec{B} = \langle 4, 2 \rangle$.
$\vec{A} + \vec{B}$, $\vec{A} - \vec{B}$ and $2\vec{A}$ are shown.

Figure 1: Adding and multiplying vectors



$\vec{A} = \langle 1.5, 2 \rangle$. $|\vec{A}| = 2.5$. $\hat{A} = \vec{A}/|A| = \langle .6, .8 \rangle$.

Length, direction, etc.

The *length* or *magnitude* of vector $\vec{V}$, denoted $|\vec{V}|$, is given by the Pythagorean theorem:

$$|\vec{V}| = \sqrt{v_1^2 + \ldots + v_k^2}$$

For example $|\langle 1,3\rangle| = \sqrt{1^2+3^2} = \sqrt{10} = 3.162$. $|\langle 2,-1,2\rangle| = \sqrt{2^2+-1^2+2^2} = \sqrt{9} = 3$.

Two important facts:

- $|p \cdot \vec{V}| = \text{abs}(p) \cdot |V|$.
- $|\vec{V} + \vec{U}| \leq |\vec{V}| + |\vec{U}|$ (the triangle inequality).

$\vec{V}$ is a *unit vector* if $|\vec{V}| = 1.$. Unit vectors are often notated with a hat rather than an arrow, e.g. $\hat{V}$.

If $\vec{V} \neq \vec{0}$ then the *direction* of $\vec{V}$ is the unit vector parallel to $\vec{V}$, which is equal to $\vec{V}/|\vec{V}|$.

If $\vec{A}$ and $\vec{B}$ are vectors, then the *distance* from $\vec{A}$ to $\vec{B}$ = the length of the line connecting their heads = $|\vec{B} - \vec{A}|$.

Dot Product

Let $\vec{A} = \langle a_1 \ldots a_k\rangle$ and $\vec{B} = \langle b_1 \ldots b_k\rangle$ be k-dimensional vectors. The *dot product* of $\vec{A}$ and $\vec{B}$, denoted $\vec{A} \cdot \vec{B}$ is a number, defined as follows

$$\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2 + \ldots + a_k b_k$$

The dot product has the following geometric interpretation: Let α be the angle between $\vec{A}$ and $\vec{B}$. Then $\vec{A} \cdot \vec{B} = |\vec{A}| \cdot |\vec{B}| \cdot \cos(\alpha)$.

A number of important properties of the dot product should be noted. Most of these are obvious consequences, either of the definition or of the above geometric formula or both.

- $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
- $(\vec{A} + \vec{B}) \cdot \vec{C} = (\vec{A} \cdot \vec{C}) + (\vec{B} \cdot \vec{C})$
- $p \cdot (\vec{A} \cdot \vec{B}) = (p \cdot \vec{A}) \cdot \vec{B}$.

- $\vec{A} \cdot \vec{A} = |\vec{A}|^2$.
- $-|\vec{A}||\vec{B}| \leq \vec{A} \cdot \vec{B} \leq |\vec{A}||\vec{B}|$.
- If $\hat{A}$ and $\hat{B}$ are unit vectors then $\hat{A} \cdot \hat{B} = \cos(\alpha)$, where α is the angle between them. Therefore, the smaller the distance from $\hat{A}$ to $\hat{B}$, the larger is $\hat{A} \cdot \hat{B}$.
- Let $\vec{A}$ and $\vec{B}$ be any two non-zero vectors. Then the angle α between $\vec{A}$ and $\vec{B}$ is given by
$$\cos(\alpha) = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}$$
- If all the components of $\vec{A}$ and $\vec{B}$ are non-negative, then $\vec{A} \cdot \vec{B}$ is non-negative.

Sparse vectors

The standard data structure used for a vector is a list or an array whose elements are the components of the vector. However, in many applications, the important vectors tend to be *sparse*; that is, most of the components are equal to 0. In that case, there are more efficient data structures. The simplest of these is a list of pairs of the form [dimension, component] for each non-zero component, sorted by dimension. For example, the vector $\langle 0, 0, 5, 0, 0, 0, -1, 2, 0, 0, 7 \rangle$ would be represented as the list (using 1-based indexing) $[[3, 5], [7, -1], [8, 2], [11, 7]]$.

It should be obvious how to implement the above operations on vectors represented in this way.