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# Book review

**Where mathematics comes from: how the embodied mind brings mathematics into Being**, edited by G. Lakoff and R. Nuñez, Basic Books, 2000, pp. 493.

In the Library of Congress cataloguing system, the call number for *Where Mathematics Comes From*, by George Lakoff and Raphael Nuñez, is QA141.I5.L37 2000, in the mathematics section. At the NYU Bookstore, the book is shelved under Philosophy. George Lakoff is a linguist; Raphael Nuñez is a cognitive psychologist. (As it happens, the author of this review is a computer scientist, specializing in artificial intelligence.) This is an unusually multi-sided enterprise.

The objective of the research programme described here is the development of a cognitive account of how people conceptualize, understand and reason about mathematics. George Lakoff is famous for his earlier linguistic work, which studied the pervasive use in language of systematic metaphorical mappings between domains. For example, difficulties are conceptualized as burdens: thus we speak of being *weighed down* with responsibility; of having a *heavy/light load* of work; of being *overburdened*; and so on. Here, Lakoff and Nuñez propose, similarly, that metaphor is the central cognitive capacity underlying the understanding of mathematics.

The theory, in brief, is as follows: There are two starting points in the human mental makeup for the development of mathematics. first is *innate arithmetic*, the arithmetic of numbers no greater than about four. Counting of groups of this size can be 'subitized'—that is, the number of such a group is immediately recognized, in any configuration—and addition and subtraction of groups of this size is immediately carried out. Moreover, there is extensive experimental evidence that these abilities are possessed, both by infants a few days old and by other creatures.[1] The second starting point is a collection of four basic experiences: dealing with collections of objects, assembling larger objects out of smaller objects, measuring lengths with measuring sticks, and moving along a path.

The development of the natural numbers and of their arithmetic takes off from these starting points. First, one observes that there is a metaphorical mapping from small instances of each of the primary experiences to innate arithmetic; e.g. that collections of size 2 correspond to the innate concept of 2, and that the operation of combining a collection of size 2 with a collection of size 1 corresponds to the innate concept of $2+1$. One then uses these metaphors to extrapolate from innate arithmetic to the arithmetic of the natural numbers. These are known as the four 'grounding metaphors'.

The remainder of mathematics is, for the most part, built up through cascaded metaphors that map one domain of mathematics to another; these are called

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'linking metaphors'. (There is also at least one additional grounding metaphor: from objects in a container to naive set theory and propositional logic.) Of these, the most extensively studied in this book is the Basic Metaphor of Infinity (BMI), which allows an infinitary mathematical structure to be conceptualized as the 'final resultant state' of an infinite sequence of operations. The central secton of this book (chapters 8–14) discuss in detail how the BMI is used to construct Cantorian infinite sets and real analysis. There are other kinds of linking metaphors, such as the 'Algebraic Essence Metaphor' used in abstract algebra, and there are additional cognitive architectural structures, such as blending two domains, but these are much less studied in the book, and I will skip over them in this review. The method used in this kind of analysis of how mathematical ideas are conceptualized is called 'mathematical idea analysis'.

The book proceeds to a discussion of the implication of this theory for the philosophy of mathematics. It ends with a four-chapter explanation and proof of Euler's formula $e^{\pi i} = -1$; this requires defining the exponential and trigonometric functions, the number $e$ and power-series expansions.

All this seems plausible and even exciting. However, my feeling is that the book in fact contains very little of any value. Specifically:

- The analysis of arithmetic in terms of grounding metaphors, though potentially an important and fruitful idea, is done extremely sloppily and needs to be entirely rethought.
- The analysis of advanced mathematics is merely re-inventing the wheel badly, rephrasing in an unclear and unworkable language definitions and concepts that mathematicians have worked very hard to formulate in a precise and usable language. There is little or no evidence for the claims that the analysis here is closer to the actual cognitive mechanisms than standard mathematical analysis. Thus, for the 'linking metaphors' between mathematical domains, 'mathematical idea analysis' is just the same kind of search for good definitions and foundations that mathematicians have been pursuing for at least the last two centuries; the only difference is that mathematicians do it more carefully.
- The philosophizing is shallow and careless and evades critical issues. The philosophical conclusions do not follow from the evidence.
- The whole book is pervaded by a contempt for classical mathematics and for mathematicians. It is assumed that mathematicians are clever but shallow symbol-mongerers, who neither know nor care what is the meaning of the symbols they manipulate. Only cognitive scientists, armed with 'mathematical idea analysis', can find out what they are talking about and explain it to them. (If this seems like an exaggeration, I'll bring quotes later.) Let me proceed to details.

## 1. Arithmetic and the grounding metaphors

The first thing one notices on carefully examining the 'grounding metaphors' for arithmetic is that Lakoff and Nuñez have 'cheated' on the source domains. That is, their description of the source domains includes entities and features that have no justification in terms of the inherent theory of the domain, but are just back constructions, introduced to support the metaphor. For example, the 'Arithmetic as Object Construction' metaphor (p. 65) posits that 'Objects [consisting of ultimate parts of unit size]' (their bracketed note) map to 'Numbers'; but actual objects are not composed out of parts of unit size (unless they have in mind a Lego set, in which case they should say so). Similarly, in the Measuring Stick metaphor (p. 68) 'Numbers' correspond to 'Physical segments [consisting of ultimate parts of unit length]', and 'One' corresponds to 'The basic physical segment'; in the 'Motion Along a Path' (p. 72) domain 'One' corresponds to 'the unit location'. None of these have any reality in the source domain, or in the cognitive theory of the source domain. Indeed, for all of these three domains, the mapping into the natural numbers is unnatural and forced; the natural mathematical model is the group of the reals under addition (for 'motion along a path') or the semi-group of positive reals (for 'composing objects' and 'measuring stick').

Another example: in the 'Object Construction' metaphor, 'Multiplication ($A \cdot B = C$)' corresponds to 'The repeated addition [A times] of A parts of size B to yield a whole object of size C;'[2] and 'Division ($C/B = A$)' corresponds to 'The repeated subtraction of parts of size B from an initial object of size C until the initial object is exhausted. The result, A, is the number of times the subtraction occurs'. But neither of these are activities that naturally occur, either in work or in play, with constructing objects; they might possibly occur in an 'educational' game specifically designed to teach arithmetic, but not, one would think, a very appealing game. The same is true of the definitions of multiplication and division in the 'Measuring stick' and 'Motion along a Path' metaphors.

There is also some sloppiness in the target domains. In the 'Arithmetic as Object Construction' metaphor, we are told that 'Objects' map to 'Numbers' and that 'The size of an object' maps to 'The size of a number'; but there is no such thing as the *size* of a number as distinct from the number itself.

On the other hand, this list of four grounding metaphors omits other basic experiences that map into numbers or into arithmetic operations. (This list does not seem to be intended as merely four examples out of many; the authors seem committed to these as the canonical '4G's'.) One is repetition over time—Da, da, da, dum $\rightarrow$ 4; a rather strange omission, since they mention in chapter 2 (p. 21) that rats are aware of this kind of numerosity. Another simple example is the interpretation of multiplication $A \cdot B$ as the number of elements in a rectangle of A by B elements. (In general, geometry tends to get short-changed in this book.) This interpretation, in contrast to the eight given in chapter 3, has the advantage that commutativity is immediately obvious.

## 2. Classes, symbolic logic and sets (chapter 6 and 7)

I omit chapter 5, which has to do with algebra. I do not find it very satisfactory, but the issues it raises are subtle and mostly irrelevant to the remainder of the book.

The main failing of chapter 6 is that it overemphasizes the importance of the 'Container' metaphor and underemphasizes the importance of the connection to predication. (The connection to predication is mentioned on p. 123, and then is dropped and does not reappear.) It is true that in *conscious* reasoning about *abstract* sets, people are apt to visualize Venn diagrams or containers. But this is far less important than the fact that Boolean logic captures some of the structure of *unconscious*, automatic, reasoning about properties. If you are told that Fred is a guppy, and you immediately infer (almost always unconsciously) that Fred is a fish and that Fred is not a dog, using the previously known facts that all guppies are fish and that no dogs are fish, there is no evidence whatever that you are doing this by visualizing a Venn diagram with regions marked 'Guppy', 'Fish' and 'Dog'. It is this kind of reasoning that is the key source for the structure of propositional logic and Boolean set theory, not the abstract 'If X is an element of A and A is disjoint from B, then X is not an element of B'.

The motivation for deemphasizing the connection with predication is obvious; these kinds of observations seriously get in the way of the claim that formal logic has no relevance to actual cognition.

In chapter 7, there is one strange and important mistake. The authors contrast their notion of 'Same Number As' with Cantor's concept of pairability, and claim that under their definition, there are not the same number of even integers as of integers. Their definition is: 'Group A has the same number of elements as group B if, for every member of A, you can take away a corresponding member of B and not have any members of B left over'. They go on to brag about how they have thus straightened out the field (p. 143):

> This distinction has never before been stated explicitly using the idea of conceptual metaphor. Indeed, because the distinction has been blurred, generations of students have been confused. Consider the following statement made by many mathematics teachers: '*Cantor proved that there are just as many positive even integers as natural numbers*'.
>
> Given our ordinary concept of 'As Many As', Cantor proved no such thing. He proved only that the sets were pairable . . . .
>
> Cantor . . . intended pairability to be a *literal* extension of our ordinary notion of Same Number As from finite to infinite sets. There Cantor was mistaken.

But, of course, their definition of 'Same Number As' is *identical* to Cantor's; their 'corresponding' is the same as his 'pairing' and the process of taking objects away is irrelevant. You can prove that there are 'As Many' even positive numbers as positive number by using the same correspondence $E = 2N$ that Cantor uses; each time you take away N from the set of integers, take away 2N from the set of even numbers.

Aside from this error, note that Lakoff and Nuñez have slipped from their self-chosen descriptive task to a prescriptive one. They do not say that this is how people *do* think about 'Same Number As' in infinite sets, and they certainly do not bring any cognitive evidence of it; they say that this is how people *should* think about it. That is, they are no longer doing cognitive science, they are doing mathematics. Not that there's anything wrong with that, except that they elsewhere say that they are definitely not telling mathematicians how to do mathematics.

This shift from descriptive to prescriptive mode occurs again explicitly in chapter 11, where the authors hawk their own theory of infinitesimals as a great improvement over both the classical theory of the reals and Robinson's theory of the hyper-reals. A little more subtly, it suffuses the entire discussion of real analysis in chapters 12, 13 and 14, where they challenge the classical formulation of real analysis and the standard interpretation of established theorems such as the space-filling curve.

## 3. The BMI and the real numbers

Chapters 8 through 14 are concerned with using the Basic Metaphor of Infinity to construct new mathematical entities and domains as limits of infinite sequence of operations over previously existing mathematical objects, with a particular focus on developing the real line and real analysis. In the process of doing that, they review this history of the 19th-century rigorization of real analysis, and take Cauchy, Dedekind and Weierstrass severely to task for having taken the definition of the real line away from its natural conceptualization.

Let us start with the first example of the BMI, defining the projective plane and the line at infinity. The following table, laying out the BMI, is reproduced from p. 168: (The left hand side is the same for all BMI's.)

| Parallel lines meet at infinity | | |
|---|---|---|
| Target domain | | Special case |
| Iterative processes that go on and on | | Projective geometry |
| The beginning state {0} | $\Longrightarrow$ | The isosceles-triangle frame with triangle $ABC_0$. |
| State {1} resulting from the initial stage of the process | $\Longrightarrow$ | Triangle $ABC_1$, where the length of $AC_1$ is $D_1$. |
| The process: From a prior intermediate state $\{n-1\}$, produce the next state $\{n\}$ | $\Longrightarrow$ | Form $AC_n$ from $AC_{n-1}$, by making $D_n$ arbitrarily longer than $D_{n-1}$. |
| The intermediate result after that iteration of the process (the relation between $n$ and $n-1$) | $\Longrightarrow$ | $D_n > D_{n-1}$ and $(90^\circ - \alpha_n) < (90^\circ - \alpha_{n-1})$. |
| 'The final resultant state' (actual infinity) | $\Longrightarrow$ | $\alpha_\infty = 90^\circ, D_\infty$ is infinitely long. Sides $AC_\infty$ and $BC_\infty$ are infinitely long parallel, and meet at $C_\infty$ a point 'at infinity' |
| Entailment E: The final resultant state ('$\infty$') is unique and follows every nonfinal state. | $\Longrightarrow$ | Entailment E: There is a unique $AC_\infty$ (distance $D_\infty$) that is longer than $AC_n$ (distance $D_n$) for all finite $n$. |

From a mathematical point of view, this is woefully inadequate. First, it omits the key question of when two such processes lead to the *same* point at infinity (whenever the bases $AB$ in the two sequences are parallel) and when they lead to different points at infinity. Second, it does not establish the properties of the points at infinity thus constructed, especially what it means for such a point to be incident on a line. Third, it does not exclude the case where the $D_n$ converge to a finite value, in which case the triangles converge to a finite triangle. None of these points is addressed in the surrounding text, either. Note that to make this work in a definition, it is not sufficient to use a sequence of isosceles triangles; you have to use an equivalence class of such sequences. But this equivalence class construction lies outside the BMI as they have defined it. Once the definition is fixed, then it is exactly the kind of definition one finds in math textbooks, e.g.

> **Definition:** Define an *isosceles triangle sequence* on base AB to be a sequence of isosceles triangles $ABC_0 \subset ABC_1 \ldots$ such that the length $AC_n$ grows without bound (i.e. for any length $L$ there exists $n$ such that $L <$ length($AC_n$). Define two such sequences, one built on the base AB and one built on the base $A'B'$ to be equivalent if AB is parallel to $A'B'$. Then a *point at infinity* is an equivalence class of isosceles triangle sequences. (... Goes on to define the line at infinity and the incidence relation)

One is unlikely to find this particular definition in the textbooks, because it is unnecessarily complex. But the point is, the construction of the BMI is just standard mathematical procedure, vaguely worded.

Well, Lakoff and Nuñez may say, formulating mathematically tight definitions is work for the mathematicians. They are cognitive scientists; their project is to determine how people think about the projective plane. But they admit here (p. 170) that there is no experimental evidence that people think about infinite sequences of isosceles triangles when they think about the projective plane. Personally, when I think about the projective plane, I imagine the line at infinity as the horizon (that is, I impose a perspective transformation that brings it into the finite plane). That could even be a grounding metaphor, grounded in the experience of railroad tracks and so on. I know, I know: subjective introspections are not scientific evidence, especially when given by a scientist with a dog in the fight. But they have no better evidence for their sequence of isosceles triangles. Someone else might think of a point at infinity as the set of all lines in a given direction, or as just one representative line in a given direction. Or, much more likely, a single person can use different modes of thinking about it depending on the problem at hand.

And why, in any case, should one think that there is a 'natural' conceptualization of the projective plane that is substantially different from the definitions in the math textbooks? Lakoff and Nuñez characterize their job as elucidating the non-technical conceptualizations of mathematical ideas. For concepts defined by grounding metaphors, that makes perfect sense; formal mathematics does not and cannot explore the connection between arithmetic and measuring sticks, say, because measuring sticks are not mathematical objects. Moreover the relation between measuring sticks and numbers is learned young, and to some extent as part of language learning; one might well suppose that there is something going on here that is not in math textbooks. But mathematics can, and does, explore the connection between the finite plane and the projective plane; and projective geometry is learned by adults in formal settings (that is, either in classrooms or from textbooks). Why should one suppose that they develop an intuitive understanding of the projective plane substantially different from the mathematical definition? It seems more likely that their understanding is an internalized form of the formal definition. And why should one suppose that all people have the same such understanding, or that a single person has only one such understanding? And why should one suppose that a cognitive scientist can simply guess at which of the many ways of constructing the projective plane happens to be that common, unconscious understanding?

To go through all the material on the BMI and the real line at this level of detail, showing where they have reinvented existing mathematical concepts using what amount to standard mathematical techniques, where their reinventions do not work, where their claims about cognitive reality are unsupported and their descriptions about standard mathematical practice off the mark, would require a very long review. I will therefore confine myself to a few points that particularly struck me.

On p. 187 they take issue with the standard definition of limits

The sequence $x_n$ has limit $L$ if for every $\epsilon > 0$ there exists $n_0$ such that for all $n > n_0$, $|x_n - L| < \epsilon$

(my wording); first, because it does not capture the idea of 'approaching' and second because this constraint includes the irrelevant case where $\epsilon = 43$. They claim, correctly, that students find the definition confusing because of the triple alternation of quantifiers. The definition they substitute (p. 195) using the BMI is much more complicated; it involves identifying a subsequence whose elements approach the limit monotonically. Even so, it involves a negated double alternation of quantifiers; the fifth step includes the constraint 'There is no positive real number $r$ such that $0 < r < |x_n - L|$ for all $x_n$ in $S$'. There is no evidence presented, and I do not find it inherently plausible, either that this is closer to an intuitive understanding of the concept, or that this is easier to learn.

They construct the real numbers in three different ways using the BMI: a real number is an infinite sequence of decimals; a real number is a least upper bound of a convergent increasing sequence of rationals; a real number is the intersection of a convergent sequence of intervals with rational bounds. The latter two are standard definitions in the mathematical literature, and their presentation here is not substantively different from the standard presentation. The first is also a standard definition; it is less preferred because it is much less elegant (each different numerical base gives, strictly speaking, a different BMI; and then there is the awkwardness about 0.9999 and 1.0000 being the same thing.) There is no reason to think that any of these are closer to an 'intuitive' understanding than Dedekind cuts, which later (chapter 13) they deplore as taking mathematics far from its intuitive foundations.

The main question I have about their analysis of the real line is, why construct it using linking metaphors, such as the BMI, at all? Why not construct it using a grounding metaphor? After all, as I mentioned above, such experiences as measuring sticks, sizes of objects, motion along a path, continuous change of all kinds, map much more naturally into the real line than into the integers. A study of the grounding metaphors for the real line would be extremely useful, and by definition outside the scope of formal mathematics. The only reason I can imagine that they haven't done this is that they *dislike* the theory of the reals, and are therefore unwilling to suppose that it is close enough to primitive experience to be the object of a grounding metaphor. (It might also be objected that the reals cannot be connected to 'subitizable' mathematics. But, as I have mentioned above, the there is no need to restrict the cognitive foundation of mathematics to subitization in the strict sense; and people certainly have some innate abilities to deal with real-valued quantities. For example, one can 'immediately' see that one stick is more than twice as long as another, if in fact it is substantially more than twice as long.)

## 4. Philosophy of mathematics

Lakoff and Nuñez derive from their analysis a subjectivist philosophy of mathematics: Mathematics is a construction of the human mind and is grounded in bodily experience; the only mathematics is embodied mathematics; there is no reason to believe that the mathematics that humans have developed has any claim to external reality or universality. To believe otherwise is to succumb to a false and dangerous 'Romance' of mathematics. At the same time, they reject a radical social constructivist view (p. 365): mathematics is not a matter of mere social agreement and is not *purely* historically and culturally contingents, because it reflects fundamental features of the human brain and fundamental experiences. This philosophical argument is developed at length in chapters 15 and 16, but pervades the rest of the book.

To my mind, this philosophizing is the weakest aspect of the book. On the other hand, philosophy is not my strong suit either, and I do not want to follow them deep into philosophical thickets. So I will confine myself to a few short points where I feel reasonably sure of my ground.

One odd thing is that the authors sometimes seem not entirely confident of their own philosophy, in the sense that they make weak claims where they could easily make much stronger ones. For example, they say (p. 131) 'There is no reason to believe that the universal class has any objective existence at all. That is, there is no scientifically valid reason to believe that the physical entities in the universe form a subclass of an objectively existing universal class'. But there is no empirical evidence to believe that *any* classes 'exist'. What empirical evidence is there, or could there possibly be, for believing that there 'exists' a class whose elements are the Eiffel Tower and the Empire State Building? Classes are theoretical constructs in any case.

The problem of why mathematics is so effective in the physical sciences is raised (p. 342) but not dealt with at all adequately. What Lakoff and Nuñez say is that there are regularities in the physical world, and that mathematical laws are designed to fit those regularities. But that does not begin to address the real issue, which is why the kinds of regularities encountered in the physical sciences fit so neatly into mathematical structures; why physics textbooks are so full of mathematical equations. There are regularities in invertebrate anatomy and in the construction of novels as well; but there are few equations in books on invertebrate anatomy and none in books on literary criticism.

Indeed, the fact that mathematics is applied to solve real-world problems is barely mentioned in this book. Admittedly, that is true of many forms of mathematical philosophy, but it seems odd in a book that takes a cognitive viewpoint, and odder still in a book that devotes several chapters to the history of the development of the calculus. One would never guess, from the discussion of the calculus in this book, that calculus was developed in order to solve problems in physics and geometry and is very effective at that. To my mind, this is reminiscent of Chomsky's dogged insistence that the use of language as a medium of communication is largely irrelevant to the study of language. This major omission, of course, makes it much easier to claim that there is nothing inevitable about the truths of mathematics.

Finally, Lakoff and Nuñez indulge themselves much too much in proclaiming universally accepted truisms as if they were daring iconoclasm, and in setting up pathetic straw men and knocking them down with loud huzzas. Sample truism: ‘Mathematical symbolism is *not* an analysis of mathematical ideas. Mathematical notation must be understood in terms of mathematical ideas’. (the ‘morals’ of chapter 5, p. 120). Sample straw man (p. 121): ‘There is a widespread belief that [symbolic logic and the logic of classes] characterize reason itself... But recent cognitive science, concerned with the embodiment of mind, has found that these branches of mathematics are far from adequate for the characterization of human reason, which must include prototypes and image schemas, as well as conceptual frames, metaphors, and blends’. As it happens, my work and that of my research community is largely devoted analysing human reason in logical terms; and even in that community of die-hard logicists, I do not know anyone who would endorse the first statement or deny the second. The most that anyone would claim is that certain types of reasoning can be described in terms of symbolic logic, such as the ‘guppy’ example I mentioned above. The belief that symbolic logic is adequate for all types of reasoning is not ‘widespread’; it is non-existent.

## 5. The proof of Euler’s formula

Chapters 17–20 contain a proof of Euler’s formula $e^{\pi i} = -1$. The proof follows a conventional form; they derive the Taylor series $e^x = \sum_n x^n/n!$, $\cos(x) = \sum_n (-1)^n x^{2n}/(2n)!$, $\sin(x) = \sum_n (-1)^n x^{2n+1}/(2n+1)!$, then show by substitution that $e^{i\theta} = \cos\theta + i\sin\theta$. To do this, they have to introduce the complex numbers, the exponential and trigonometric functions, and the Taylor series; since their purpose inherently demands that they include a lot of explanation and motivation, it is not surprising that this takes 70 pages.

There are gaps in the proof and at least one error. The error is the claim (p. 444) that the Taylor series for a continuous real-valued function always converges to the function; this is not true, even if the function is required to be continuous and infinitely differentiable. Gaps include:

- They do not define $a^x$ where $x$ is real. They posit that it must satisfy the laws of exponents, but do not show that any function satisfying the laws of exponents actually exists.
- They use the fact that $d(\sin x)/dx = \cos x$ and $d(\cos x)/dx = -\sin x$, but do not prove these.

However, a more serious problem with this proof, in terms of their objectives, is that a proof by manipulating Taylor series is *not* one that gives any insight; quite the contrary, it is very much a symbol crunching proof. There are more interesting proofs of Euler’s theorem. For instance: begin by showing that $e^x = \lim_{n\to\infty}(1 + x/n)^n$; then show that multiplication by $(1 + i/n)$ corresponds to rotation in the complex plane by $1/n$ radians, to first-order in $1/n$; thus $e^{i\theta}$ corresponds to rotation in the complex plane by $\theta$ radians[3]. Another type of proof works directly with the differential equations, without resorting to the Taylor series. A deeper, though more difficult proof, is to show that the definition $e^{a+bi} = e^a(\cos(b) + i\sin(b))$ is the only extension of $e^x$ to the complex plane that is differentiable in the complex plane.

But, to my mind, none of these proofs entirely dispel the mystery of this great formula. To some extent, the famous quote of Benjamin Peirce (quoted here on p. 383)—'Gentlemen,that is surely true, it is absolutely paradoxical, we cannot understand it,and we do not know what it means. But we have proved it, and therefore we know it is true'. —still stands. Certainly, the analysis in this book does nothing to dispel it.

## 6. Contempt for mathematicians

I am sure the authors do not intend this; some of their best friends are mathematicians. Nonetheless, the view throughout this book is that mathematicians neither know nor care what is the meaning of what they are doing.

A good example is in their introduction to the analysis of Euler's formula (p. 383):

> Relatively few mathematics teachers understand [Euler's formula] even today, and fewer students do. Yet generation after generation of mathematics teachers and students continue to go uncomprehendingly through one version or another of Euler's proof, understanding only the regularity in the manipulation of the symbols.
>
> They are much like Mr. M., Laurent Cohen and Stanislas Dehaene's patient discussed in Chapter 1, who know that 'three times nine is twenty-seven' but not what it means.
>
> Mr. M., being brain-damaged has no choice. Benjamin Peirce was born too soon. But in the age of cognitive science one can at least try to do better.

It is not true. It is not even close to true. And if comparing all but 'relatively few' mathematicians to a brain-damaged patient does not qualify as contempt, I do not know what would. The fact is that very few competent mathematician will find either new content or new insight in Lakoff and Nuñez's explanation of the proof. And any math major who has taken a course in function of a complex variable understands Euler's formula in a much deeper sense than can be attained here, because he or she will understand how the formula fits within the general theory of the extension of analytic functions to the complex plane. Certainly there would have been nothing new to Peirce; all of these approaches to conceptualizing the issue were well established in his time.

## 7. What should be pursued

In my opinion, what is worth pursuing in this research programme is the study of the grounding metaphors (though I am not sure that the 'metaphor' construction is as central as the authors believe); that is, the connection between real-world domains and the related mathematics. For example:

- As discussed above, the real line corresponds to experiences of 'real-valued' quantities, such as lengths, weights, durations, and so on.
- Real-valued functions correspond to the experience of continuous change.

- Euclidean geometry corresponds to the experience of perceiving and interacting with space.
- Projective geometry corresponds to the experience of perspective.
- Probability theory corresponds to the experiences of uncertainty and of chance events.

If the cognitive structures involved in understanding real-world domains are much more carefully examined, and more domains and types of mathematics are considered, this could yield results that are well-established, valuable and exciting. I expect that what will be found is that *every* real-world application of mathematics is potentially a grounding metaphor; that is, a source of insight into the associated mathematics.

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**Endnotes**

[1] Chapter 2 of this book surveys this work. An excellent extensive treatment can be found in *The Number Sense: How the Mind Creates Mathematics*, by Stanislas Dehaene (Oxford University Press, 1997). It is not clear why Lakoff and Nuñez focus so strictly on subitization; as discussed at length in Dehaene, both humans and other animals have the innate ability to perform mathematical operations such as comparison over numbers much larger than four, though with some loss of precision.

[2] The use of number $A$ here to mean something other than a size of an object is an instance of 'blending'.

[3] My brother Frank showed this to me some years ago. I have not seen it in print.