A lower bound for the root separation of polynomials

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April 21, 2008

Abstract

An experimental study by Collins (JSC; 2001) suggested the conjecture that the minimum separation of real zeros of irreducible integer polynomials is about the square root of Mahler's bound for general integer polynomials. We prove that a power of about two thirds of the Mahler bound is already a lower bound for the minimum root separation of all integer polynomials.

Keywords: Polynomial roots, minimum root separation, Taylor series, resultants.

1 Introduction

The minimum root separation is the fundamental measure for verified inclusions of zeros of polynomial systems via algebraic algorithms, see, e.g., [8, 11]. It is an important tool for classification of transcendental numbers, see [1] and the references cited therein.

Definition 1.1 *The* minimum root separation *of an integer polynomial P given as*

$$P(x) := \sum_{i=0}^{n} a_i x^i = a_n \cdot \prod_{i=1}^{n} (x - \zeta_i), \ \textit{where} \ a_n \neq 0,$$

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is defined as

$$sep(P) := \min_{\zeta_i \neq \zeta_j} |\zeta_i - \zeta_j|.$$

We call $M(P) := |a_n| \prod_{i=1}^{n} \max\{1; |\zeta_i|\}$ *the* Mahler measure *of* P.

The size *of* P*, denoted by* $s(P)$*, or* s *for short, is defined as*

$$s(P) := \sum_{i=0}^{n} |a_i|.$$

In formulating estimates for the minimum root separation, we capture the case of a single, n-fold zero of P by considering the separation in this case as $sep(P) = +\infty$.

Mahler's root separation estimate for $P(x) = \sum a_i x^i \in \mathbb{Z}[x]$ may be formulated in terms of the coefficient vector norms $\|P\|_q := \|(a_0, \ldots, a_n)\|_q; q = 1, 2$ using the fact that

$$M(P) \leq (\sum |a_i|^2)^{1/2} \leq \sum |a_i| = s \tag{1}$$

(where the first inequality follows easily from Jensen's inequality *viz.* [4] or [9]). The best known estimate for the minimum root separation was obtained by Mahler [5] in 1964.

Theorem 1.1 *Let* $P(x) = \sum_0^n a_i x^i$ *be an integer polynomial of size* s *and degree* n*. Then*

$$sep(P) > \frac{\sqrt{3} \cdot \sqrt{discr(P)}}{n^{n/2+1} \cdot M(P)^{n-1}} \geq \frac{\sqrt{3} \cdot \sqrt{discr(P)}}{n^{n/2+1} \cdot s^{n-1}}. \tag{2}$$

This yields a trivial estimate in case of a polynomial with multiple zeros as the discriminant vanishes, but Mahler's estimate may be applied to the square-free integer polynomial $\hat{P}(x) := P(x)/gcd(P(x), P'(x))$ with $sep(\hat{P}) = sep(P)$, and $discr(\hat{P}) \geq 1$. The relation $M(\hat{P}) \leq M(P) \leq s$ holds true, and thus (2) gives rise to the following *general* estimate.

Corollary 1.1 *Let* $P(x) = \sum_0^n a_i x^i$ *be an integer polynomial of size* s *and degree* n*. Then*

$$sep(P) > \frac{\sqrt{3}}{n^{n/2+1} \cdot s^{n-1}}. \tag{3}$$

Do there exist polynomials with *small* root separation? The example by Bugeaud and Mignotte [1] from 2004,

$$P(X) := (X^n - aX + 1)^k - 2X^{nk-k}(aX - 1)^k,\ n \geq 3, k \geq 2, a \geq 10, \quad (4)$$

has a cluster of k zeros inside a circle with radius $2a^{-2n}$ centered at $1/a + 1/a^{n+1}$. This shows that the separation might decrease with the size s like $1/s^{n/2}$. Which is the best possible exponent of s in (3)?

We claim that a power of *about two third* of the lower bound (3) is a lower bound for the minimum root separation.

Theorem 1.2 *Let $P(x) = \sum_0^n a_i x^i$ be an integer polynomial of size s and degree n. Then*

$$sep(P) > \frac{1}{4e \cdot 2^{n/3} \cdot n^{n/3+2} \cdot (s+1)^{2n/3}}. \quad (5)$$

We proof this theorem in Section 3. Our interest was sparked by a conjecture of Collins [3] supposing that for real zeros of irreducible integer polynomials the square root of Mahler's general bound (3) might be a lower bound for the minimum root separation. This conjecture has to be taken *cum grano salis* as it is well known that for cubic polynomials the exponent of s in the root separation bound is precisely $-2 = -n/2 - 1/2$, cf., e.g., [9].

2 Bounding values of polynomials via resultants

To prove estimates for the minimum root separation nearly all authors used manipulations of specially constructed resultants or the discriminant (like Cauchy [2] and Mahler [5]). The only exception known to us is the work of S.M. Rump [8] who considered the Taylor expansion of P at a root ξ of the derivative P'. In the following, we sketch the elements necessary to estimate the function value $P(\xi)$.

The norms $\|\cdot\|_q$ may be extended to matrices of polynomials by considering an $n \times n$ square matrix as a one-dimensional vector of length n^2. The following generalization of Hadamard's lemma is easy, for a proof see [11].

Lemma 2.1 *Let $M = (m_{ij})_{i,j=1,\ldots,n}$ be a quadratic matrix over $\mathbb{C}[X]$, i.e. $m_{ij} \in \mathbb{C}[X]$. Then*

$$\|det(M)\|_2 \leq \prod_{i=1}^{n} \left(\sum_{j=1}^{n} \|m_{ij}\|_1^2 \right)^{1/2}.$$

This allows to proof the following estimate (variant of a result in [8]) for the value of integer polynomials at algebraic numbers.

Lemma 2.2 *Let P and Q be arbitrary non-constant integer polynomials. Suppose for some $\beta \in \mathbf{C}$ that*

$$P(\beta) \neq 0, \ \textit{but} \quad Q(\beta) = 0.$$

Then it holds with $m = deg(P), n = deq(Q), f = s(P), g = s(Q)$:

$$|P(\beta)| \geq ((f+1)^n \cdot g^m)^{-1}.$$

Proof: We consider the resultant $r(y) := res_x(P(x) - y, Q(x)) \in \mathbb{Z}[y]$ of degree n which is defined as the determinant of the Sylvester matrix of $P(x) - y$ and $Q(x)$ considered as polynomials of x. The coefficient vector of $P(x) - y$ has 1-norm $f + 1$. The generalized Hadamard Lemma yields $\|r\|_2 \leq (f + 1)^n \cdot g^m$. We consider the reciprocal polynomial r^* of r (defined as $r^*(y) = y^n r(1/y)$) with norm $\|r^*\|_2 = \|r\|_2$ and root $1/P(\beta)$. From (1) we get the estimate $\max\{1; |1/P(\beta)|\} \leq \|r^*\|_2$ which yields the lower bound. □

For a zero of the derivative (i.e. a *critical point*) the Lemma yields the following *viz.* [8].

Proposition 2.1 *Let P be an arbitrary integer polynomial of size s and degree $n \geq 2$. If ξ satisfies*

$$P'(\xi) = 0, but \quad P(\xi) \neq 0,$$

then

$$|P(\xi)| \geq (s^n n^n \cdot (s+1)^{n-1})^{-1} > (n^n \cdot (s+1)^{2n-1})^{-1}. \tag{6}$$

This holds true even if P and P' have common roots.

3 Taylor series and root separation

The Grace-Heawood Lemma (cf. [6], Th.23.1) and its generalisations (cf. [6], Chap. 6) show that the distance of a critical point to a root is bounded in terms of the root separation. We use the following result essentially due to Marden.

Lemma 3.1 *Let P be an arbitrary complex polynomial of degree $n \geq 2$. Let $P(\alpha) = P(\beta) = 0$, and $\alpha \neq \beta$. Then there exists some critical point γ with $P'(\gamma) = 0 \neq P(\gamma)$ and*

$$\max\{|\gamma - \alpha|; |\gamma - \beta|\} \quad \leq \quad csc(\frac{\pi}{2n-2}) \cdot |\beta - \alpha| < 2n \cdot |\beta - \alpha|. \qquad (7)$$

Proof: The first inequality in (7) is a consequence of [6], Th.25.4 as α, β lie in a circle of radius $R := |\alpha - \beta|/2$ centered at $C := (\alpha + \beta)/2$. Marden's result [6], Th.25.4 implies existence of a γ as above with distance to C at most $csc(\frac{\pi}{2n-2})R$, and the first inequality follows easily. The second inequality follows after further easy calculations with the trigonometric function, see [8]. □

Proof of Theorem 1.2: In the remainder of the paper let P be a polynomial which has at least two distinct zeros, and minimal root separation for given $s \geq 2$ and $n \geq 2$. We may restrict our analysis to polynomials with $sep(P) \leq (4e)^{-1}/(n^2 \cdot s)$, because otherwise the claimed estimate (5) trivially holds true.

With a suitable numbering suppose that $|\zeta_1 - \zeta_2| = sep(P)$. Obviously,

$$\min\{|\zeta_1|; |\zeta_2|\} \leq 1,$$

because otherwise the roots $1/\zeta_1$ and $1/\zeta_2$ of $P^*(x) = x^n P(1/x)$ had separation $|\frac{\zeta_2 - \zeta_1}{\zeta_1 \zeta_2}| < |\zeta_1 - \zeta_2|$ contradicting the choice of P.

By Lemma 3.1 we may choose a zero ξ of P' (i.e. a critical point of P) unequal to either of ζ_1 and ζ_2, and with distance to these zeros not exceeding $2n|\zeta_1 - \zeta_2|$. We estimate $|\xi|$ as

$$|\xi| \leq |\zeta_1| + |\xi - \zeta_1| \leq |\zeta_1| + 2n|\zeta_2 - \zeta_1| \leq 1 + \frac{2n}{4en^2 s} < 1 + \frac{1}{n}. \qquad (8)$$

We write

$$\zeta_1 - \xi = -h, \ \zeta_2 - \xi = h + \epsilon, \tag{9}$$

$$\zeta_2 - \zeta_1 = 2h + \epsilon, \ \text{ for some} \, \epsilon \in \mathbb{C}, \tag{10}$$

$$\text{which implies} \qquad 2n \cdot sep(P) = 2n|\zeta_2 - \zeta_1| \geq \max\{|h|; |h+\epsilon|\}. \tag{11}$$

The Taylor series for $P(\zeta_1)$ at ξ yields the relation

$$-P(\xi) = \frac{h^2}{2} P''(\xi) + \sum_{i=3}^{n} \frac{(-h)^i}{i!} P^{(i)}(\xi). \tag{12}$$

If $l > 1$ is the smallest index such that $P^{(l)}(\xi) \neq 0$, we also consider the Taylor expansion of $P(\zeta_1) = 0$ around ξ with complex Lagrange remainder term according to Darboux (for ref., see [10], p.96) as

$$-\frac{(-h)^l}{l!} P^{(l)}(\xi) = P(\xi) + \omega \frac{(-h)^{l+1}}{(l+1)!} P^{(l+1)}(\xi + t(\zeta_1 - \xi)), |\omega| \leq 1, 0 \leq t \leq 1. \tag{13}$$

We may suppose that

$$|h| \leq \frac{1}{2(s+1)^{2n/3} n^{n/3+1}}, \tag{14}$$

because otherwise by (11) $2n \cdot sep(P) \geq |h| > \frac{1}{2(s+1)^{2n/3} n^{n/3+1}}$, and (5) already holds true.

To prove the estimate (5) we distinguish three cases.

Case 1: $P'(\xi) = 0 = P(\xi)$.

The root ξ is close to the roots ζ_1, ζ_2, and we may re-use Mahler's proof [5] of Theorem 1.1 (see, e.g., [11]) as follows. Let us consider $\hat{P} := P/gcd(P, P')$ which retains the roots ζ_1, ζ_2, ξ, but has a non-vanishing discriminant, and $M(\hat{P}) \leq M(P)$. Denote ξ by ζ_3. Suppose that the degree of $\hat{P}$ is m.

To make the analysis slightly more general (as needed below) we treat the situation of three distinct zeros such that

$$\hat{P}(\zeta_3) = \hat{P}(\zeta_1) = \hat{P}(\zeta_2) = 0, \, deg(\hat{P}) = m, \text{and} \, |\zeta_3 - \zeta_1| \leq 20n^2 |\zeta_1 - \zeta_2|. \tag{15}$$

As the discriminant of $\hat{P}$ is non-zero, we use the well-known identity (see, e.g., [11])

$$discr(\hat{P}) = \left[a_n^{n-1} \det(\zeta_l^k)_{\substack{k=0,\ldots,m-1\\ l=0,\ldots,m-1}}\right]^2$$

of the discriminant with the scaled square of the determinant of the Vandermonde matrix. We modify Mahler's analysis (as in the proof of the Mahler-Davenport bound, see, e.g., [11]) to find

$$1 \le \sqrt{discr(\hat{P})} \le |\zeta_1 - \zeta_2| \cdot |\zeta_1 - \zeta_3| M(\hat{P})^{m-1} (m^{1.5}/\sqrt{3})^2 m^{\frac{m-2}{2}}.$$

By (15), $|\zeta_3 - \zeta_1| \le 20n^2 |\zeta_1 - \zeta_2| = 20n^2 sep(P)$, and using this together with $m < n$ yields

$$sep(P) \ge (\sqrt{20/3} s^{n/2-0.5} n^{n/4+1.5})^{-1}. \tag{16}$$

Case 2: $P'(\xi) = 0 \ne P(\xi)$ and $P''(\xi) = 0$.

The modulus of any $P^{(l)}(\lambda)$ may be estimated as

$$|P^{(l)}(\lambda)| \le n^l \sum_{i=l}^{n} |a_i| \max\{1; |\lambda|\}^{i-l} \le n^l \cdot s \cdot \max\{1; |\lambda|\}^n. \tag{17}$$

With $P''(\xi) = 0$, the smallest $l > 1$ with $P^{(l)}(\xi) = 0$ is at least 3. Using (13) for $\lambda = \xi$ with $|\xi|$ estimated as (8) and $|h|$ restricted by (14) we obtain

$$\begin{aligned} |h^l| &\ge \left|\frac{P(\xi)}{P^{(l)}(\xi)/l!}\right| - \left|\frac{h^{l+1}}{(l+1)!} \frac{P^{(l+1)}(\xi + t(\zeta_1 - \xi))}{P^{(l)}(\xi)/l!}\right| \\ &\ge \frac{l!}{n^l se} \left(|P(\xi)| - \left(\frac{1}{2(s+1)^{2n/3} n^{n/3+1}}\right)^{l+1} \frac{n^{l+1} s \cdot e}{(l+1)!} \right). \end{aligned}$$

As l is at least 3, and $|P(\xi)|$ is at least $(s+1)^{-(2n-1)} n^{-n}$ by (6), we obtain

$$2n \cdot sep(P) \ge |h| \ge \frac{1}{2} \frac{|P(\xi)|^{1/3}}{n s^{1/3} e^{1/3}} > \frac{1}{2e^{1/3}} \frac{1}{(s+1)^{2n/3} n^{n/3+1}}.$$

Case 3: $P'(\xi) = 0 \ne P(\xi) \cdot P''(\xi)$.

Let us consider first the favourable situation that P'' is small, more precisely, limited by

$$|P''(\xi)| \leq 5es \cdot n^4 2^n sep(P). \tag{18}$$

The smallest index $l > 1$ such that $P^{(l)}(\xi) \neq 0$ is precisely 2. We use the Taylor expansion with complex remainder (13), and re-write it to estimate

$$|h^2/2| \geq |\frac{P(\xi)}{P''(\xi)}| - |\frac{h^3}{6}\frac{P^{(3)}(\xi + t(\zeta_1 - \xi))}{P''(\xi)}| \text{ for some } t, 0 \leq t \leq 1.$$

The limitation (14) for $|h|$ together with (6) yields the inequality

$$\begin{aligned} |h^2| &\geq \frac{2}{|P''(\xi)|}\left(|P(\xi)| - (\frac{1}{2(s+1)^{2n/3}n^{n/3+1}})^3 \frac{n^3 s \cdot e}{6}\right) \\ &\geq \frac{2}{|P''(\xi)|}\frac{1}{2(s+1)^{2n-1}n^n}. \end{aligned}$$

We use the assumed upper limit (18) for $|P''(\xi)|$ together with the upper bound (11) for $|h|$ to obtain

$$\begin{aligned} (2n \cdot sep(P))^2 \geq |h^2| &\geq \frac{1}{|P''(\xi)|}\frac{2}{2(s+1)^{2n-1}n^n} \\ &\geq \frac{2}{5esn^4 2^n sep(P)}\frac{1}{(s+1)^{2n-1}n^n}. \end{aligned}$$

This yields the estimate $sep(P) > (20^{1/3}e^{1/3}2^{n/3}(s+1)^{2n/3}n^{n/3+2})^{-1}$, and our claim (5) holds true if (18) is satisfied.

If one of the roots ζ_1, ζ_2 is not simple, $|P''(\xi)|$ must be small: A multiple root of P is a root of the derivative. Thus, we have the situation that P' has two distinct roots, namely ξ and at least one of ζ_1, ζ_2, in a distance of at most $2n|\zeta_1 - \zeta_2|$. We may assume w.l.o.g. that the multiple root is ζ_2. If we write with a suitable numbering $P'(x) = na_n \prod_{i=1}^{n-1}(x - \lambda_i) = na_n(x - \xi)(x - \zeta_2)\prod_{j=3}^{n-1}(x - \lambda_j)$, then $P''(\xi) = na_n(\xi - \zeta_2)\prod_{j=3}^{n-1}(\xi - \lambda_j)$. The Mahler measure of P' is $M(P') = |n \cdot a_n| \prod_{i=1}^{n-1} \max\{1; |\lambda_i|\}$, and as the coefficient vector of P' has 1-norm at most $n \cdot s$, inequality (1) yields

$M(P') \leq n \cdot s$. This implies the estimate

$$\begin{aligned}
|P''(\xi)| &= |na_n||\xi - \zeta_2| \prod_{j=3}^{n-1} |\xi - \lambda_j| \\
&\leq |na_n| 2n |\zeta_1 - \zeta_2| \max\{1; |\xi|\}^{n-2} \prod_{j=3}^{n-1} 1 + \max\{1; |\lambda_j|\} \\
&\leq 2nsep(P) \max\{1; |\xi|\}^{n-2} 2^{n-2} M(P') \\
&< 2^{n-1} n^2 se \cdot sep(P).
\end{aligned}$$

This puts us in the situation (18), and hence our claimed estimate holds true if one of the roots ζ_1, ζ_2 of minimal separation is not simple.

Let us deal with the favourable situation that a third root is close to the roots ζ_1, ζ_2 measured in terms of distance to ξ. If

$$\min_{3 \leq j \leq n} |\xi - \zeta_j| \leq 10n \cdot |h| \leq 20n^2 |\zeta_1 - \zeta_2|$$

we have the situation of (15) (with $m \leq n$), and the lower bound (16).

Thus, we may suppose that the roots ζ_1 and ζ_2 are simple, and all other roots are somewhat remote from ζ_1, ζ_2:

$$\min_{3 \leq j \leq n} |\xi - \zeta_j| > 10n \cdot |h| = 10n |\zeta_1 - \xi|. \tag{19}$$

We distinguish two different geometrical situations: The mid-point $\frac{\zeta_1 + \zeta_2}{2}$ is far from ξ, or otherwise close to it measured in terms of h. We quantify the latter situation (using the differences $\zeta_1 - \xi = -h$, $\zeta_2 - \xi = h + \epsilon$ and their sum $\epsilon = \zeta_1 + \zeta_2 - 2\xi$) as

$$|\epsilon| = |\zeta_1 + \zeta_2 - 2\xi| < |h|/10.$$

We write $P(\xi)$ in terms of h and ϵ

$$P(\xi)/a_n = \prod_{i=1}^{n} (\xi - \zeta_i) = -h(h + \epsilon) \prod_{\nu=3}^{n} (\xi - \zeta_\nu) =: D, \tag{20}$$

and want to do the same with

$$P''(\xi)/a_n = \sum_{j=1}^{n} \sum_{\substack{k=1 \\ k\neq j}}^{n} \prod_{\substack{\nu=1 \\ \nu\neq j,k}}^{n} (\xi - \zeta_\nu) =: N.$$

We may divide this double sum into parts (using the usual conventions about empty sums and empty products):

$$\begin{aligned} N = \prod_{\nu=3}^{n}(\xi-\zeta_\nu) \; &+ (\xi-\zeta_1)\sum_{k=3}^{n}\prod_{\substack{\nu=3 \\ \nu\neq k}}^{n}(\xi-\zeta_\nu) \\ &+ (\xi-\zeta_2)\sum_{k=3}^{n}\prod_{\substack{\nu=3 \\ \nu\neq k}}^{n}(\xi-\zeta_\nu) \\ &+ (\xi-\zeta_1)(\xi-\zeta_2)\sum_{j=3}^{n}\sum_{\substack{k=3 \\ k\neq j}}^{n}\prod_{\substack{\nu=3 \\ \nu\neq j,k}}^{n}(\xi-\zeta_\nu). \end{aligned}$$

As $P(\xi) \neq 0$, the ratio N/D is well-defined, and we may write

$$\begin{aligned} \frac{P''(\xi)}{P(\xi)} = \frac{N}{D} &= \frac{1}{-h(h+\epsilon)}\left[1 - \epsilon\sum_{k=3}^{n}\frac{1}{\xi-\zeta_k} - h(h+\epsilon)\sum_{j=3}^{n}\sum_{\substack{k=3 \\ k\neq j}}^{n}\frac{1}{\xi-\zeta_j}\frac{1}{\xi-\zeta_k}\right] \\ &=: \frac{1}{-h(h+\epsilon)}B. \end{aligned}$$

Expressing alternatively $-\frac{P(\xi)}{P''(\xi)} = -D/N$ via the Taylor series expansion (12) yields

$$\begin{aligned} -\frac{P(\xi)}{P''(\xi)} = \frac{-D}{N} &= \frac{h^2}{2} + \sum_{i=3}^{n}\frac{(-h)^i}{i!}\frac{P^{(i)}(\xi)}{P''(\xi)} \\ &= h^2\Big(\frac{1}{2} + \sum_{i=3}^{n}\frac{(-1)^i h^{i-2}}{i!}\frac{P^{(i)}(\xi)}{P''(\xi)}\Big) =: h^2\cdot T. \end{aligned}$$

Thus, we may consider the identity

$$B\cdot T = \frac{h(h+\epsilon)}{h^2}.$$

We estimate the bracket B (using (19) and $|\epsilon| < |h|/10$), and obtain

$$\left(1 + \frac{|h|}{10}\sum_{k=3}^{n}\frac{1}{10n|h|} + \frac{11}{10}\frac{|h|^2(n-3)^2}{(10n|h|)^2}\right)\cdot|\frac{1}{2} + \sum_{i=3}^{n}\frac{(-1)^i h^{i-2}}{i!}\frac{P^{(i)}(\xi)}{P''(\xi)}| \geq \frac{9}{10}$$

which implies

$$|\frac{1}{2} + \sum_{i=3}^{n}\frac{(-1)^i h^{i-2}}{i!}\frac{P^{(i)}(\xi)}{P''(\xi)}| \geq \frac{2}{3}.$$

The triangle inequality together with the estimate for the derivatives yields

$$\frac{1}{6} \leq \sum_{i=3}^{n}|\frac{h^{i-2}}{i!}\frac{P^{(i)}(\xi)}{P''(\xi)}| \leq \sum_{i=3}^{n}|\frac{h^{i-2}}{i!}\frac{e\cdot n^i\cdot s}{P''(\xi)}|,$$

and as the limitation (14) for $|h|$ implies $|h| < 1/(2n)$ we have

$$|P''(\xi)| \leq 6es(\frac{n^3}{3!} + \frac{h\cdot n^4}{4!} + \frac{h^2\cdot n^5}{5!} + \ldots) < 2|h|es\cdot n^3 \leq 4esn^4\cdot sep(P).$$

This puts us in the situation of (18), and hence our claimed new estimate (5) holds true in the case $|\epsilon| < |h|/10$.

It remains to consider the case $|\epsilon| \geq |h|/10$. We may restrict our analysis to the case $sep(P) < \frac{1}{2(s+1)^{2n/3}n^{n/3+2}}$ as otherwise the claimed separation bound is trivially true. We repeat our notation and introduce additionally z_1, z_2:

$$z_1 := \zeta_1 - \xi = -h,\; z_2 := \zeta_2 - \xi = h + \epsilon, \text{ for some } \epsilon \in \mathbb{C}.$$

We define Blaschke factors $B_{z_i}(x) := \dfrac{1 - x\overline{z_i}}{x - z_i}$; $i = 1, 2$. For $|x| = 1$, the factors $B_{z_i}(x)$ are unimodular. We compose the holomorphic function

$$f(x) := P(x+\xi)B_{z_1}(x)B_{z_2}(x)$$

to which we want to apply Cauchy's inequality (cf., e.g., [10], p.91) for the Taylor coefficients in the form

$$\sup_i|\frac{f^{(i)}(0)}{i!}| \leq \max_{|x|\leq 1}\{|f(x)|\} =: M_f. \tag{21}$$

Using the bound (8) for $|\xi|$ we may estimate the maximum M_f as

$$M_f = \max_{|x|\le 1}\{|P(x+\xi)|\} \le (1+|\xi|)^n \max_{|x|\le 1}\{|P(x)|\} \le 2^n e\cdot s \tag{22}$$

(where the first inequality follows from Hadamard's three circle theorem, cf., e.g., [7]).

Expanding P around the critical point ξ (where $P'(\xi)=0$) yields $P(x+\xi) = p_0 + p_2x^2 + \sum_{i=3}^n p_i x^i$. For the Taylor coefficients of the composite function f we obtain from (21) and (22) the relation

$$|f'(0)| = |p_0\cdot(B'_{z_1}(0)B_{z_2}(0) + B_{z_1}(0)B'_{z_2}(0))| \le 2^n es.$$

We write explicitly

$$|p_0\cdot(\frac{1-|z_1|^2}{z_1^2}\frac{1}{z_2} + \frac{1}{z_1}\frac{1-|z_2|^2}{z_2^2})| \le 2^n es,$$

and deduce

$$2^n s\cdot e \ge |p_0\cdot\frac{z_1+z_2}{z_1^2z_2^2}| - |p_0\overline{\frac{z_1+z_2}{z_1z_2}}|. \tag{23}$$

As $p_0 = P(\xi) = a_n\prod_{i=1}^n(\xi-\zeta_i), z_1 = -h = \zeta_1-\xi, z_2 = h+\epsilon = \zeta_2-\xi$ and $z_1+z_2=\epsilon$ we have

$$\begin{aligned}|p_0\overline{\frac{z_1+z_2}{z_1z_2}}| = |\epsilon|\cdot|a_n\prod_{j=3}^n(\xi-\zeta_j)| &\le |\epsilon|\cdot\max\{1;|\xi|\}^{n-2}2^{n-2}|a_n|\prod_{j=3}^n\max\{1;|\zeta_j|\}\\ &\le |\epsilon|\cdot e\cdot 2^{n-2}\cdot s.\end{aligned}$$

The assumed upper limits for $|h|$ and $sep(P)$ imply by (11) that $|\epsilon| < 1/2^{n+2}$ With $|z_1+z_2| = |\epsilon| \ge |h|/10 = |z_1|/10$ and $|\epsilon| < \frac{1}{2^n}$ the inequality (23) yields

$$2^n\cdot 1.2se \ge |\frac{p_0}{10\cdot z_1z_2^2}| = |\frac{P(\xi)}{10\cdot h(h+\epsilon)^2}|.$$

If we use in this inequality the bounds (11) together with the estimate of $|P(\xi)|$ in (6) we obtain

$$sep(P)^3 \ge |\frac{P(\xi)}{2^n\cdot 10\cdot(2n)^3 1.2se}| > \frac{1}{96\cdot e\cdot 2^n\cdot n^{n+3}(s+1)^{2n}},$$

hence even in the case that $|\epsilon| \ge |h|/10$ our claimed estimate (5) holds true. Q.E.D.

References

[1] Bugeaud, Y.; Mignotte, M. On the distance between roots of integer polynomials. *Proc. Edinburgh Mathematical Society*, 47:553–556, 2004.

[2] Cauchy, A.L. *Oeuvres complètes d'Augustin Cauchy*, volume 3 of 2^e *série.*, chapter Sur la résolution numérique des équations, pages 378–426. Gauthier-Villars, Paris, reprint of 1897 edition, 1921.

[3] Collins, G.E. Polynomial minimum root separation. *Journal of Symbolic Computation*, 32:467–473, 2001.

[4] Landau, E. Sur quelques théorèmes de *M. Pétrovitch* relatifs aux zéros des fonctions analytiques. *Bull. Soc. Math. Franc.*, 33:251–261, 1905.

[5] Mahler, K. An Inequality for the Discriminant of a Polynomial. *Michigan Math. Journal*, 11:257–262, 1964.

[6] Marden, M. M. *The Geometry of Polynomials.* AMS, Providence, Rhode Island, second edition, 1966.

[7] Pólya, G.; Szegö, G. *Aufgaben und Lehrsätze aus der Analysis I.* Springer-Verlag, Berlin, 1970.

[8] Rump, S.M. Polynomial minimum root separation. *Mathematics of Computation*, 33:327–336, 1979.

[9] Schönhage, A. Polynomial root separation examples. *Journal of Symbolic Computation*, 41(10):1080–1090, 2006.

[10] Whittaker, E. T.; Watson, G. N. *Modern Analysis.* Cambridge University Press, 4th edition, 1927.

[11] Yap, C.K. *Fundamental Problems of Algorithmic Algebra.* Oxford University Press, New York, N.Y., 2000.