A lower bound for the root separation of polynomials

P. Batra*

April 21, 2008

Abstract

An experimental study by Collins (JSC; 2001) suggested the conjecture that the minimum separation of real zeros of irreducible integer polynomials is about the square root of Mahler’s bound for general integer polynomials. We prove that a power of about two thirds of the Mahler bound is already a lower bound for the minimum root separation of all integer polynomials.

Keywords: Polynomial roots, minimum root separation, Taylor series, resultants.

1 Introduction

The minimum root separation is the fundamental measure for verified inclusions of zeros of polynomial systems via algebraic algorithms, see, e.g., [8, 11]. It is an important tool for classification of transcendental numbers, see [1] and the references cited therein.

Definition 1.1 The minimum root separation of an integer polynomial $P$ given as

$$P(x) := \sum_{i=0}^{n} a_i x^i = a_n \cdot \prod_{i=1}^{n} (x - \zeta_i), \text{ where } a_n \neq 0,$$

*Hamburg University of Technology, Inst. f. Computer Technology, 21071 Hamburg, Germany (batra@tuhh.de). [Phone: ++49(40)42878-3478, Fax: -2798]
is defined as

\[
\text{sep}(P) := \min_{\zeta_i \neq \zeta_j} |\zeta_i - \zeta_j|.
\]

We call \( M(P) := |a_n| \prod_{i=1}^{n} \max\{1; |\zeta_i|\} \) the Mahler measure of \( P \).

The size of \( P \), denoted by \( s(P) \), or \( s \) for short, is defined as

\[
s(P) := \sum_{i=0}^{n} |a_i|.
\]

In formulating estimates for the minimum root separation, we capture the case of a single, \( n \)-fold zero of \( P \) by considering the separation in this case as \( \text{sep}(P) = +\infty \).

Mahler’s root separation estimate for \( P(x) = \sum a_i x^i \in \mathbb{Z}[x] \) may be formulated in terms of the coefficient vector norms \( \|P\|_q := \|(a_0, \ldots, a_n)\|_q \); \( q = 1, 2 \) using the fact that

\[
M(P) \leq (\sum |a_i|^2)^{1/2} \leq \sum |a_i| = s
\]

(where the first inequality follows easily from Jensen’s inequality viz. [4] or [9]). The best known estimate for the minimum root separation was obtained by Mahler [5] in 1964.

**Theorem 1.1** Let \( P(x) = \sum a_i x^i \) be an integer polynomial of size \( s \) and degree \( n \). Then

\[
\text{sep}(P) > \frac{\sqrt{3} \cdot \sqrt{\text{discr}(P)}}{n^{n/2+1} \cdot M(P)^{n-1}} \geq \frac{\sqrt{3} \cdot \sqrt{\text{discr}(P)}}{n^{n/2+1} \cdot s^{n-1}}.
\]

This yields a trivial estimate in case of a polynomial with multiple zeros as the discriminant vanishes, but Mahler’s estimate may be applied to the square-free integer polynomial \( \hat{P}(x) := P(x)/\gcd(P(x), P'(x)) \) with \( \text{sep}(\hat{P}) = \text{sep}(P) \), and \( \text{discr}(\hat{P}) \geq 1 \). The relation \( M(\hat{P}) \leq M(P) \leq s \) holds true, and thus (2) gives rise to the following general estimate.

**Corollary 1.1** Let \( P(x) = \sum a_i x^i \) be an integer polynomial of size \( s \) and degree \( n \). Then

\[
\text{sep}(P) > \frac{\sqrt{3}}{n^{n/2+1} \cdot s^{n-1}}.
\]
Do there exist polynomials with small root separation? The example by Bugeaud and Mignotte [1] from 2004,

\[ P(X) := (X^n - aX + 1)^k - 2X^{nk-k}(aX - 1)^k, \quad n \geq 3, k \geq 2, a \geq 10, \quad (4) \]

has a cluster of \( k \) zeros inside a circle with radius \( 2a^{-2n} \) centered at \( 1/a + 1/a^{n+1} \). This shows that the separation might decrease with the size \( s \) like \( 1/s^{n/2} \). Which is the best possible exponent of \( s \) in (3)?

We claim that a power of about two third of the lower bound (3) is a lower bound for the minimum root separation.

**Theorem 1.2** Let \( P(x) = \sum_0^n a_i x^i \) be an integer polynomial of size \( s \) and degree \( n \). Then

\[ \text{sep}(P) > \frac{1}{4e \cdot 2^{n/3} \cdot n^{n/3+2} \cdot (s + 1)^{2n/3}}. \quad (5) \]

We proof this theorem in Section 3. Our interest was sparked by a conjecture of Collins [3] supposing that for real zeros of irreducible integer polynomials the square root of Mahler’s general bound (3) might be a lower bound for the minimum root separation. This conjecture has to be taken cum grano salis as it is well known that for cubic polynomials the exponent of \( s \) in the root separation bound is precisely \(-2 = -n/2 - 1/2\), cf., e.g., [9].

## 2 Bounding values of polynomials via resultants

To prove estimates for the minimum root separation nearly all authors used manipulations of specially constructed resultants or the discriminant (like Cauchy [2] and Mahler [5]). The only exception known to us is the work of S.M. Rump [8] who considered the Taylor expansion of \( P \) at a root \( \xi \) of the derivative \( P' \). In the following, we sketch the elements necessary to estimate the function value \( P(\xi) \).

The norms \( \| \cdot \|_q \) may be extended to matrices of polynomials by considering an \( n \times n \) square matrix as a one-dimensional vector of length \( n^2 \). The following generalization of Hadamard’s lemma is easy, for a proof see [11].
Lemma 2.1 Let $M = (m_{ij})_{i,j=1,...,n}$ be a quadratic matrix over $\mathbb{C}[X]$, i.e. $m_{ij} \in \mathbb{C}[X]$. Then
\[
\|\det(M)\|_2 \leq \prod_{i=1}^{n} \left( \sum_{j=1}^{n} \|m_{ij}\|_1^2 \right)^{1/2}.
\]
This allows to proof the following estimate (variant of a result in [8]) for the value of integer polynomials at algebraic numbers.

Lemma 2.2 Let $P$ and $Q$ be arbitrary non-constant integer polynomials. Suppose for some $\beta \in \mathbb{C}$ that $P(\beta) \neq 0$, but $Q(\beta) = 0$.
Then it holds with $m = \deg(P)$, $n = \deg(Q)$, $f = s(P)$, $g = s(Q)$:
\[
|P(\beta)| \geq ((f + 1)^n \cdot g^m)^{-1}.
\]

Proof: We consider the resultant $r(y) := \text{res}_x(P(x) - y, Q(x)) \in \mathbb{Z}[y]$ of degree $n$ which is defined as the determinant of the Sylvester matrix of $P(x) - y$ and $Q(x)$ considered as polynomials of $x$. The coefficient vector of $P(x) - y$ has 1-norm $f + 1$. The generalized Hadamard Lemma yields $\|r\|_2 \leq (f + 1)^n \cdot g^m$. We consider the reciprocal polynomial $r^*$ of $r$ (defined as $r^*(y) = y^n r(1/y)$) with norm $\|r^*\|_2 = \|r\|_2$ and root $1/P(\beta)$. From (1) we get the estimate $\max\{1; |1/P(\beta)|\} \leq \|r^*\|_2$ which yields the lower bound. For a zero of the derivative (i.e. a critical point) the Lemma yields the following viz. [8].

Proposition 2.1 Let $P$ be an arbitrary integer polynomial of size $s$ and degree $n \geq 2$. If $\xi$ satisfies
\[
P'(\xi) = 0, \text{ but } P(\xi) \neq 0,
\]
then
\[
|P(\xi)| \geq (s^n n^n \cdot (s + 1)^{n-1})^{-1} > (n^n \cdot (s + 1)^{2n-1})^{-1}.
\]
This holds true even if $P$ and $P'$ have common roots.
3 Taylor series and root separation

The Grace-Heawood Lemma (cf. [6], Th.23.1) and its generalisations (cf. [6], Chap. 6) show that the distance of a critical point to a root is bounded in terms of the root separation. We use the following result essentially due to Marden.

Lemma 3.1 Let $P$ be an arbitrary complex polynomial of degree $n \geq 2$. Let $P(\alpha) = P(\beta) = 0$, and $\alpha \neq \beta$. Then there exists some critical point $\gamma$ with $P'(\gamma) = 0 \neq P(\gamma)$ and

$$\max\{|\gamma - \alpha|; |\gamma - \beta|\} \leq \csc\left(\frac{\pi}{2n-2}\right) \cdot |\beta - \alpha| < 2n \cdot |\beta - \alpha|. \quad (7)$$

Proof: The first inequality in (7) is a consequence of [6], Th.25.4 as $\alpha, \beta$ lie in a circle of radius $R := |\alpha - \beta|/2$ centered at $C := (\alpha + \beta)/2$. Marden’s result [6], Th.25.4 implies existence of a $\gamma$ as above with distance to $C$ at most $\csc\left(\frac{\pi}{2n-2}\right)R$, and the first inequality follows easily. The second inequality follows after further easy calculations with the trigonometric function, see [8]. □

Proof of Theorem 1.2: In the remainder of the paper let $P$ be a polynomial which has at least two distinct zeros, and minimal root separation for given $s \geq 2$ and $n \geq 2$. We may restrict our analysis to polynomials with $sep(P) \leq (4e)^{-1}/(n^2 \cdot s)$, because otherwise the claimed estimate (5) trivially holds true.

With a suitable numbering suppose that $|\zeta_1 - \zeta_2| = sep(P)$. Obviously,

$$\min\{|\zeta_1|; |\zeta_2|\} \leq 1,$$

because otherwise the roots $1/\zeta_1$ and $1/\zeta_2$ of $P^*(x) = x^n P(1/x)$ had separation $|\zeta_1 - \zeta_2| < |\zeta_1 - \zeta_2|$ contradicting the choice of $P$.

By Lemma 3.1 we may choose a zero $\xi$ of $P'$ (i.e. a critical point of $P$) unequal to either of $\zeta_1$ and $\zeta_2$, and with distance to these zeros not exceeding $2n|\zeta_1 - \zeta_2|$. We estimate $|\xi|$ as

$$|\xi| \leq |\zeta_1| + |\xi - \zeta_1| \leq |\zeta_1| + 2n|\zeta_2 - \zeta_1| \leq 1 + \frac{2n}{4en^2s} < 1 + \frac{1}{n}. \quad (8)$$
We write
\[
\zeta_1 - \xi = -h, \; \zeta_2 - \xi = h + \epsilon, \quad (9)
\]
\[
\zeta_2 - \zeta_1 = 2h + \epsilon, \quad \text{for some } \epsilon \in \mathbb{C}, \quad (10)
\]
which implies
\[
2n \cdot \text{sep}(P) = 2n|\zeta_2 - \zeta_1| \geq \max\{|h|; |h + \epsilon|\}. \quad (11)
\]
The Taylor series for \( P(\zeta_1) \) at \( \xi \) yields the relation
\[
-P(\xi) = \frac{h^2}{2} P''(\xi) + \sum_{i=3}^{n} \frac{(-h)^i}{i!} P^{(i)}(\xi). \quad (12)
\]
If \( l > 1 \) is the smallest index such that \( P^{(l)}(\xi) \neq 0 \), we also consider the Taylor expansion of \( P(\zeta_1) = 0 \) around \( \xi \) with complex Lagrange remainder term according to Darboux (for ref., see [10], p.96) as
\[
-\frac{(-h)^l}{l!} P^{(l)}(\xi) = P(\xi) + \omega \frac{(-h)^{l+1}}{(l+1)!} P^{(l+1)}(\xi + t(\zeta_1 - \xi)), \quad |\omega| \leq 1, 0 \leq t \leq 1. \quad (13)
\]
We may suppose that
\[
|h| \leq \frac{1}{2(s+1)^{2n/3}n^{n/3+1}}, \quad (14)
\]
because otherwise by (11) \( 2n \cdot \text{sep}(P) \geq |h| > \frac{1}{2(s+1)^{2n/3}n^{n/3+1}} \), and (5) already holds true.

To prove the estimate (5) we distinguish three cases.

Case 1: \( P'(\xi) = 0 = P(\xi) \).

The root \( \xi \) is close to the roots \( \zeta_1, \zeta_2 \), and we may re-use Mahler’s proof [5] of Theorem 1.1 (see, e.g., [11]) as follows. Let us consider \( \hat{P} := P/gcd(P, P') \) which retains the roots \( \zeta_1, \zeta_2, \xi \), but has a non-vanishing discriminant, and \( M(\hat{P}) \leq M(P) \). Denote \( \xi \) by \( \zeta_3 \). Suppose that the degree of \( \hat{P} \) is \( m \).

To make the analysis slightly more general (as needed below) we treat the situation of three distinct zeros such that
\[
\hat{P}(\zeta_3) = \hat{P}(\zeta_1) = \hat{P}(\zeta_2) = 0, \quad \text{deg}(\hat{P}) = m, \quad \text{and} \quad |\zeta_3 - \zeta_1| \leq 20n^2|\zeta_1 - \zeta_2|. \quad (15)
\]
As the discriminant of \( \hat{P} \) is non-zero, we use the well-known identity (see, e.g., [11])

\[
\text{discr}(\hat{P}) = \left[a_n^{-1} \det(\zeta^k_{i=0,...,m-1})\right]^2
\]

of the discriminant with the scaled square of the determinant of the Vandermonde matrix. We modify Mahler’s analysis (as in the proof of the Mahler-Davenport bound, see, e.g., [11]) to find

\[
1 \leq \sqrt{\text{discr}(\hat{P})} \leq |\zeta_1 - \zeta_2| \cdot |\zeta_1 - \zeta_3| M(\hat{P})^{m-1}(m^{1.5}/\sqrt{3})^2 m^{m-2}.
\]

By (15), \(|\zeta_3 - \zeta_1| \leq 20n^2|\zeta_1 - \zeta_2| = 20n^2 \text{sep}(P)\), and using this together with \(m < n\) yields

\[
\text{sep}(P) \geq (\sqrt{20/3} s^{n/2 - 0.5} n^{n/4 + 1.5})^{-1}.
\]

Case 2: \(P'(\xi) = 0 \neq P(\xi)\) and \(P''(\xi) = 0\).

The modulus of any \(P^{(l)}(\lambda)\) may be estimated as

\[
|P^{(l)}(\lambda)| \leq n^l \sum_{i=1}^n |a_i| \max\{1; |\lambda|\}^{i-l} \leq n^l \cdot s \cdot \max\{1; |\lambda|\}^n.
\]

With \(P''(\xi) = 0\), the smallest \(l > 1\) with \(P^{(l)}(\xi) = 0\) is at least 3. Using (13) for \(\lambda = \xi\) with \(|\xi|\) estimated as (8) and \(|h|\) restricted by (14) we obtain

\[
|h|^l \geq \frac{|P(\xi)|}{P^{(l)}(\xi)/l!} - \frac{h^{l+1}}{(l+1)!} \frac{P^{(l+1)}(\xi + t(\zeta_1 - \xi))}{P^{(l)}(\xi)/l!}
\]

\[
\geq \frac{ll!}{n^l \text{sep} \left(\frac{|P(\xi)|}{2(s + 1)^{2n^3/3} n^{n/3 + 1}}\right)}
\]

As \(l\) is at least 3, and \(|P(\xi)|\) is at least \((s + 1)^{-2n^3} n^{-n}\) by (6), we obtain

\[
2n \cdot \text{sep}(P) \geq |h| \geq \frac{1}{2} \frac{|P(\xi)|^{1/3}}{n^{s^{1/3} e^{1/3}}} > \frac{1}{2} \frac{1}{2^{e^{1/3}} (s + 1)^{2n^3 n^{n/3 + 1}}}
\]

Case 3: \(P'(\xi) = 0 \neq P(\xi) \cdot P''(\xi)\).
Let us consider first the favourable situation that $P''$ is small, more precisely, limited by

$$|P''(\xi)| \leq 5es \cdot n^4 \cdot 2^n \cdot \text{sep}(P). \quad (18)$$

The smallest index $l > 1$ such that $P^{(l)}(\xi) \neq 0$ is precisely 2. We use the Taylor expansion with complex remainder (13), and re-write it to estimate

$$|h^2/2| \geq \left| \frac{P(\xi)}{P'(\xi)} - \frac{h^3 P^{(3)}(\xi + t(\zeta_1 - \xi))}{6 P''(\xi)} \right| \quad \text{for some } t, 0 \leq t \leq 1.$$

The limitation (14) for $|h|$ together with (6) yields the inequality

$$|h^2| \geq \frac{2}{|P''(\xi)|} \left( |P(\xi)| - \frac{1}{2(s + 1)^{2n/3} n^{n/3+1}} \right) \frac{n^3 s \cdot e}{6} \geq \frac{2}{|P''(\xi)|} \frac{1}{2(s + 1)^{2n-1} n^n}.$$

We use the assumed upper limit (18) for $|P''(\xi)|$ together with the upper bound (11) for $|h|$ to obtain

$$(2n \cdot \text{sep}(P))^2 \geq |h^2| \geq \frac{2}{|P''(\xi)|} \frac{1}{2(s + 1)^{2n-1} n^n} \geq 2 \frac{1}{5esn^4 \cdot 2^n \cdot \text{sep}(P) (s + 1)^{2n-1} n^n}.$$ 

This yields the estimate $\text{sep}(P) > (20^{1/3} e^{1/3} 2^{n/3} (s + 1)^{2n/3} n^{n/3+2})^{-1}$, and our claim (5) holds true if (18) is satisfied.

If one of the roots $\zeta_1, \zeta_2$ is not simple, $|P''(\xi)|$ must be small: A multiple root of $P$ is a root of the derivative. Thus, we have the situation that $P'$ has two distinct roots, namely $\xi$ and at least one of $\zeta_1, \zeta_2$, in a distance of at most $2n|\zeta_1 - \zeta_2|$. We may assume w.l.o.g. that the multiple root is $\zeta_2$. If we write with a suitable numbering $P'(x) = na_n \prod_{i=1}^{n-1} (x - \lambda_i) = na_n(x - \xi)(x - \zeta_2) \prod_{j=3}^{n-1} (x - \lambda_j)$, then $P''(\xi) = na_n(\xi - \zeta_2) \prod_{j=3}^{n-1} (\xi - \lambda_j)$.

The Mahler measure of $P'$ is $M(P') = |n \cdot a_n| \prod_{i=1}^{n-1} \max\{1; |\lambda_i|\}$, and as the coefficient vector of $P'$ has 1-norm at most $n \cdot s$, inequality (1) yields
\[ M(P') \leq n \cdot s. \] This implies the estimate

\[ |P''(\xi)| = |na_n| |\xi - \zeta_2| \prod_{j=3}^{n-1} |\xi - \lambda_j| \]

\[ \leq |na_n| 2n|\zeta_1 - \zeta_2| \max\{1; |\xi|\}^{n-2} \prod_{j=3}^{n-1} 1 + \max\{1; |\lambda_j|\} \]

\[ \leq 2nse\max\{1; |\xi|\}^{n-2} 2^{n-2} M(P') \]

\[ < 2^{n-1} n^2 \cdot se \cdot sep(P). \]

This puts us in the situation (18), and hence our claimed estimate holds true if one of the roots \( \zeta_1, \zeta_2 \) of minimal separation is not simple.

Let us deal with the favourable situation that a third root is close to the roots \( \zeta_1, \zeta_2 \) measured in terms of distance to \( \xi \). If

\[ \min_{3 \leq j \leq n} |\xi - \zeta_j| \leq 10n \cdot |h| \leq 20n^2 \max|\zeta_1 - \zeta_2| \]

we have the situation of (15) (with \( m \leq n \)), and the lower bound (16).

Thus, we may suppose that the roots \( \zeta_1 \) and \( \zeta_2 \) are simple, and all other roots are somewhat remote from \( \zeta_1, \zeta_2 \):

\[ \min_{3 \leq j \leq n} |\xi - \zeta_j| > 10n \cdot |h| = 10n|\zeta_1 - \xi|. \] (19)

We distinguish two different geometrical situations: The mid-point \( \frac{\zeta_1 + \zeta_2}{2} \) is far from \( \xi \), or otherwise close to it measured in terms of \( h \). We quantify the latter situation (using the differences \( \zeta_1 - \xi = -h, \zeta_2 - \xi = h + \epsilon \) and their sum \( \epsilon = \zeta_1 + \zeta_2 - 2\xi \)) as

\[ |\epsilon| = |\zeta_1 + \zeta_2 - 2\xi| < |h|/10. \]

We write \( P(\xi) \) in terms of \( h \) and \( \epsilon \)

\[ P(\xi)/a_n = \prod_{i=1}^{n} (\xi - \zeta_i) = -h(h + \epsilon) \prod_{\nu=3}^{n} (\xi - \zeta_\nu) =: D, \] (20)
and want to do the same with
\[
P''(\xi)/a_n = \sum_{j=1}^{n} \sum_{k=1}^{n} \prod_{\nu=1, \nu \neq j, k}^{n} (\xi - \zeta_{\nu}) =: N.
\]

We may divide this double sum into parts (using the usual conventions about empty sums and empty products):
\[
N = \prod_{\nu=3}^{n} (\xi - \zeta_{\nu}) + (\xi - \zeta_1) \sum_{k=3}^{n} \prod_{\nu=3, \nu \neq k}^{n} (\xi - \zeta_{\nu})
+ (\xi - \zeta_2) \sum_{k=3}^{n} \prod_{\nu=3, \nu \neq k}^{n} (\xi - \zeta_{\nu})
+ (\xi - \zeta_1)(\xi - \zeta_2) \sum_{j=3}^{n} \sum_{k=3, k \neq j}^{n} \prod_{\nu=3, \nu \neq j, k}^{n} (\xi - \zeta_{\nu}).
\]

As \(P(\xi) \neq 0\), the ratio \(N/D\) is well-defined, and we may write
\[
\frac{P''(\xi)}{P(\xi)} = \frac{N}{D} = \frac{1}{-h(h + \epsilon)} \left[ 1 - \epsilon \sum_{k=3}^{n} \frac{1}{\xi - \zeta_k} - h(h + \epsilon) \sum_{j=3}^{n} \sum_{k=3, k \neq j}^{n} \frac{1}{\xi - \zeta_j} \frac{1}{\xi - \zeta_k} \right]
= : \frac{1}{-h(h + \epsilon)} B.
\]

Expressing alternatively \(-\frac{P(\xi)}{P''(\xi)} = -D/N\) via the Taylor series expansion (12) yields
\[
\frac{P(\xi)}{P''(\xi)} = \frac{-D}{N} = \frac{h^2}{2} + \sum_{i=3}^{n} \frac{(-h)^i}{i!} \frac{P^{(i)}(\xi)}{P''(\xi)}
= h^2 \left( \frac{1}{2} + \sum_{i=3}^{n} \frac{(-1)^i h^{i-2}}{i!} \frac{P^{(i)}(\xi)}{P''(\xi)} \right) =: h^2 \cdot T.
\]

Thus, we may consider the identity
\[
B \cdot T = \frac{h(h + \epsilon)}{h^2}.
\]
We estimate the bracket $B$ (using (19) and $|\epsilon| < |h|/10$), and obtain

$$
\left( 1 + \frac{|h|}{10} \sum_{k=3}^{n} \frac{1}{10n|h|} + \frac{11}{10} \frac{|h|^2(n - 3)^2}{(10n|h|)^2} \right) \cdot \frac{1}{2} + \sum_{i=3}^{n} \frac{(-1)^i h^{i-2} P^{(i)}(\xi)}{i! P''(\xi)} \geq \frac{9}{10}
$$

which implies

$$
\left| \frac{1}{2} + \sum_{i=3}^{n} \frac{(-1)^i h^{i-2} P^{(i)}(\xi)}{i! P''(\xi)} \right| \geq \frac{2}{3}.
$$

The triangle inequality together with the estimate for the derivatives yields

$$
\frac{1}{6} \leq \sum_{i=3}^{n} \frac{h^{i-2} P^{(i)}(\xi)}{i! P''(\xi)} \leq \sum_{i=3}^{n} \frac{h^{i-2} e \cdot n^i \cdot s}{i! P''(\xi)},
$$

and as the limitation (14) for $|h|$ implies $|h| < 1/(2n)$ we have

$$
|P''(\xi)| \leq 6es \left( \frac{n^3}{3!} + \frac{h \cdot n^4}{4!} + \frac{h^2 \cdot n^5}{5!} + \ldots \right) < 2|h|es \cdot n^3 \leq 4esn^4 \cdot sep(P).
$$

This puts us in the situation of (18), and hence our claimed new estimate (5) holds true in the case $|\epsilon| < |h|/10$.

It remains to consider the case $|\epsilon| \geq |h|/10$. We may restrict our analysis to the case $sep(P) < \frac{1}{2((s+1)^{n+1}/n^n + s^{n+1})}$ as otherwise the claimed separation bound is trivially true. We repeat our notation and introduce additionally $z_1, z_2$:

$$
z_1 := \zeta_1 - \xi = -h, \ z_2 := \zeta_2 - \xi = h + \epsilon, \text{ for some } \epsilon \in \mathbb{C}.
$$

We define Blaschke factors $B_{z_i}(x) := \frac{1 - xz_i}{x - z_i}$; $i = 1, 2$. For $|x| = 1$, the factors $B_{z_i}(x)$ are unimodular. We compose the holomorphic function

$$
f(x) := P(x + \xi)B_{z_1}(x)B_{z_2}(x)
$$

to which we want to apply Cauchy’s inequality (cf., e.g., [10], p.91) for the Taylor coefficients in the form

$$
\sup_i \left| \frac{f^{(i)}(0)}{i!} \right| \leq \max_{|x| \leq 1} \{|f(x)|\} =: M_f.
$$

(21)
Using the bound (8) for $|\xi|$ we may estimate the maximum $M_f$ as

$$M_f = \max_{|x| \leq 1} |P(x + \xi)| \leq (1 + |\xi|)^n \max_{|x| \leq 1} |P(x)| \leq 2^n e \cdot s \quad (22)$$

(where the first inequality follows from Hadamard’s three circle theorem, cf., e.g., [7]).

Expanding $P$ around the critical point $\xi$ (where $P'(\xi) = 0$) yields $P(x + \xi) = p_0 + p_2 x^2 + \sum_{i=3}^n p_i x^i$. For the Taylor coefficients of the composite function $f$ we obtain from (21) and (22) the relation

$$|f'(0)| = |p_0 \cdot (B_{z_1}(0) B_{z_2}(0) + B_{z_1}(0) B'_{z_2}(0))| \leq 2^n e s.$$  

We write explicitly

$$|p_0 \cdot (\frac{1 - |z_1|^2}{z_1^2} \frac{1}{z_2} + \frac{1 - |z_2|^2}{z_2^2})| \leq 2^n e s,$$

and deduce

$$2^n s \cdot e \geq |p_0 \cdot \frac{z_1 + z_2}{z_1^2 z_2^2} - |p_0 \cdot \frac{z_1 + z_2}{z_1 z_2}|. \quad (23)$$

As $p_0 = P(\xi) = a_n \prod_{i=1}^n (\xi - \zeta_i)$, $z_1 = -h = \xi_1 - \xi$, $z_2 = h = \xi_2 - \xi$ and $z_1 + z_2 = \epsilon$ we have

$$|p_0 \cdot \frac{z_1 + z_2}{z_1 z_2}| = |\epsilon| \cdot |a_n \prod_{j=3}^n (\xi - \zeta_j)| \leq |\epsilon| \cdot \max\{1; |\xi|\}^{n-2} 2^{n-2}|a_n| \prod_{j=3}^n \max\{1; |\zeta_j|\}$$

$$\leq |\epsilon| \cdot e \cdot 2^{n-2} \cdot s.$$  

The assumed upper limits for $|h|$ and $sep(P)$ imply by (11) that $|\epsilon| < 1/2^{n+2}$

With $|z_1 + z_2| = |\epsilon| \geq |h|/10 = |z_1|/10$ and $|\epsilon| < 1/2^n$ the inequality (23) yields

$$2^n \cdot 1.2 s e \geq \frac{|p_0 |}{10 \cdot z_1 z_2} = |\frac{P(\xi)}{10 \cdot h(h+\epsilon)^2}|.$$ 

If we use in this inequality the bounds (11) together with the estimate of $|P(\xi)|$ in (6) we obtain

$$sep(P)^3 \geq \left|\frac{P(\xi)}{2^n \cdot 10 \cdot (2n)^3 1.2 s e} \right| > \frac{1}{96 \cdot e \cdot 2^n \cdot n^{n+3}(s+1)^{2n}}.$$ 

hence even in the case that $|\epsilon| \geq |h|/10$ our claimed estimate (5) holds true.

Q.E.D.
References


