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Randomized Zero Testing of Radical Expressions and Elementary Geometry Theorem Proving

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Abstract

We develop a probabilistic test for the vanishing of *radical expressions*, that is, expressions involving the four rational operations $(+, -, \times, \div)$ and square root extraction. This extends the well-known Schwartz's probabilistic test for the vanishing of polynomials. The probabilistic test forms the basis of a new theorem prover for conjectures about ruler & compass constructions. Our implementation uses the `Core Library` which can perform exact comparison for radical expressions. Some experimental results are presented.

1 Introduction

Several approaches to proving theorems in Elementary Geometry using constructive methods in Computer Algebra were proposed in the 1980s [6]. These were much more successful than earlier approaches based on purely logical or axiomatic approaches. Thus, Kutzler and Stifter [14] and Kapur [12] proposed methods based on Gröbner bases. Carrá and Gallo [1] and Gallo [7] devised a method using the dimension underlying the algebraic variety. Hong [10] introduced semi-numerical methods ("proof by example" techniques) based on gap theorems. An acclaimed approach in this area is due to Wu [18, 20, 19] who applied the concept of characteristic sets to geometric theorem proving. Extensive experimentation with Wu's method were reported by Chou [3, 5].

All these algebraic approaches begin by translating the geometric statements into algebraic ones. A proposed geometry theorem (also called a *conjecture*) is translated algebraically into two parts: a system H of multivariate polynomials called the *hypothesis*, and a single polynomial T called the *thesis*. The conjecture is true if the vanishing of the hypothesis system implies the vanishing of the thesis polynomial. From the viewpoint of algebraic geometry, proving the conjecture amounts to showing that $Var(H) \subseteq Var(T)$ where $Var(S)$ is the algebraic variety defined by a set S of polynomials. This basic formulation must be refined in order to handle degeneracy conditions.

Wu's "basic method" computes the pseudo-remainder of the polynomial thesis with respect to the Wu-Ritt extended characteristic set of the hypotheses system. If the pseudo-remainder is zero, then the conjecture is true. Wu's basic method has been successfully used to prove many classical

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and some new theorems in plane analytic geometry. The basic method fails if the variety $Var(H)$ is reducible. To handle this, Wu's "complete method" begins by decomposing $Var(H)$ into irreducible components and applying the basic method to each component. A drawback in Wu's method is that it works with an algebraically closed field. In particular, it is not a complete method for the real algebraic varieties. The present paper addresses a special case of real algebraic varieties.

Gröbner bases methods can be doubly exponential in the worst case [16, 21]. The complexity for Wu's method is somewhat better but remains an issue. To circumvent the high complexity, we investigate probabilistic methods [11] combined with "proof by example" techniques [10]. In probabilistic theorem proving, we do not prove the validity of a conjecture in the classical sense. Instead, we either prove the invalidity of a conjecture (by showing a counter example) or else classify the conjecture as "true with the high probability $1 - \varepsilon$". This latter classification must be properly understood since, classically, it is nonsense to say that a theorem is true with some probability. What is meant is that, relative to a set of experiments we conduct, the probability that the conjecture is false *and* we failed to discover this, is less than ε.

An interesting approach along these lines was given by Carrá, Gallo and Gennaro [2]. They applied the Schwartz-Zippel [11, 24] probabilistic test for the vanishing of pseudo-remainders in Wu's method. They considered conjectures in the classical setting of *ruler & compass constructions*. Such conjectures are examined by testing the vanishing of Wu's pseudo-remainder for randomly chosen examples. Each example is specified by a random choice of values for its parameters. The random choices come from some suitable *test set* whose cardinality depends on the degree of the pseudo-remainder. The characteristic set as well as the pseudo remainder are computed. If the pseudo-remainder is zero, then the example is successful; otherwise, as in Wu's method, further investigation is called for. While implementing their method, one of us (D.T.) discovered a serious efficiency issue. The degree of the pseudo-remainder is very high: if the conjecture involves C ruler & compass construction steps, then, the degree of the pseudo-remainder in [2] (following [8, 9]) has the following bound:

$$D = 2^{O(C^3)} C^{O(C^2)}$$

The cardinality of the test set is $2D$, which is too large in practice. This bound applies to the test for "generic truth". For "universal truth", D can be improved to $2P \cdot 3^{C+1}$ where P is the number of points in the construction. Unfortunately, practically no classical theorems are universal truths.

SUMMARY OF NEW RESULTS. (1) We develop an extension of the Schwartz-Zippel probabilistic zero test. While the Schwartz-Zippel test is applicable to polynomials, we treat radical expressions by admitting the additional operations of division and square-roots. This adds considerable complexity to the proofs. Furthermore, for efficiency considerations, we use straight line programs to represent radical expressions. The asymptotic time complexity of our probabilistic test is a low-order polynomial. Since radical expressions are common in many applications, we expect this new test to be generally useful.

(2) We address the problem of computer proofs of geometric conjectures about ruler & compass constructions. The zero test of radical expressions is tailor fitted for this problem. Moreover, we combine randomness with the numerical approach of Hong to give additional efficiency. Thus, our approach appears to be intrinsically more efficient than previous general approaches (e.g., Wu's or Gröbner bases).

(3) Our prover is implemented using the `Core library` [13, 17]. This is an unexpected application of our library, which was designed as a general `C++`-package to support the Exact Geometric Computation [23, 22] approach to robust algorithms. Preliminary experimental results are quite

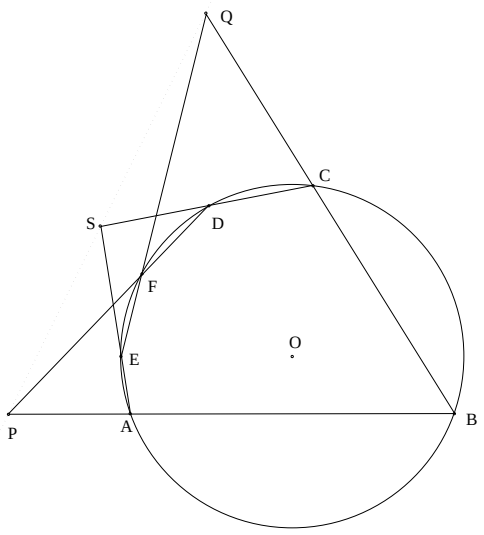
promising. We expect further improvements by fine-tuning our library for this specific application. Our prover is currently distributed with version 1.3 of the `Core library` (Aug 15, 2000) and available from `http://cs.nyu.edu/exact/core/`.

OVERVIEW. The paper is organized as follows: Section 2 gives an overview of geometric conjectures about ruler & compass constructions. Section 3 gives our extension of Schwartz's probabilistic test to radical expressions. Section 4 addresses the application of our new probabilistic test to theorem proving. We conclude in Section 5.

2 Theorem Proving for Ruler & Compass Constructions

We follow the algebraic approach which has been well-summarized by Chou [5]. Ruler & compass operations may be seen as constructing lengths, points, lines and circles, collectively called *geometric objects*. A collection of such geometric objects will be called a *geometric scene*. We consider geometric scenes that are constructed incrementally using ruler & compass operations. The algebraic analogue of constructing a geometric object O amounts to introducing a pair of variables (x, y) and corresponding polynomial equations $h_i(x, y, z, \ldots)$ $(i = 1, 2, \ldots)$ that must be satisfied if (x, y) lies on O. Here, h_i may involve other variables $z, \ldots$, from previously constructed objects. We shall classify the *variables* introduced by our constructions into two sorts: *independent* and *dependent* variables. For short, the independent variables will be called *parameters*. It is instructive to give a concrete example (from [5]).

Example 1 (Pascal's Theorem) Let A, B, C, D, F and E be six points on a circle centered at O. Let $P = AB \bigcap DF$, $Q = BC \bigcap FE$ and $S = CD \bigcap EA$. Show that P, Q and S are collinear.



Let $A = (0, 0)$, $O = (u_1, 0)$, $B = (x_1, u_2)$, $C = (x_2, u_3)$, $D = (x_3, u_4)$, $F = (x_4, u_5)$, $E = (x_5, u_6)$, $P = (x_7, x_6)$, $Q = (x_9, x_8)$, and $S = (x_{11}, x_{10})$. This gives the following equations for the hypotheses.

Equation	*Geometry*	*Remark*
$h_1 : x_1^2 - 2u_1x_1 + u_2^2 = 0$	$[OA \equiv OB]$	Introduces x_1, u_2
$h_2 : x_2^2 - 2u_1x_2 + u_3^2 = 0$	$[OA \equiv OC]$	Introduces x_2, u_3
$h_3 : x_3^2 - 2u_1x_3 + u_4^2 = 0$	$[OA \equiv OD]$	Introduces x_3, u_4
$h_4 : x_4^2 - 2u_1x_4 + u_5^2 = 0$	$[OA \equiv OF]$	Introduces x_4, u_5
$h_5 : x_5^2 - 2u_1x_5 + u_6^2 = 0$	$[OA \equiv OE]$	Introduces x_5, u_6
$h_6 : (u_5 - u_4)x_7 + (-x_4 + x_3)x_6 + u_4x_4 - u_5x_3 = 0$	$[P \in DF]$	Introduces x_6, x_7
$h_7 : u_2x_7 - x_1x_6 = 0$	$[P \in AB]$	Constrains x_6, x_7
$h_8 : (u_6 - u_5)x_9 + (-x_5 + x_4)x_8 + u_5x_5 - u_6x_4 = 0$	$[Q \in FE]$	Introduces x_8, x_9
$h_9 : (u_3 - u_2)x_9 + (-x_2 + x_1)x_8 + u_2x_2 - u_3x_1 = 0$	$[Q \in BC]$	Constrains x_8, x_9
$h_{10} : u_6x_{11} - x_5x_{10} = 0$	$[S \in AE]$	Introduces x_{10}, x_{11}
$h_{11} : (u_4 - u_3)x_{11} + (-x_3 + x_2)x_{10} + u_3x_3 - u_4x_2 = 0$	$[S \in CD]$	Constrains x_{10}, x_{11}

The conclusion that P, Q, S are collinear can be translated into the following polynomial:

$$g = (x_8 - x_6)x_{11} + (-x_9 + x_7)x_{10} + x_6x_9 - x_7x_8 = 0.$$

■

In general, we obtain system of polynomial equations, $h_1 = h_2 = \cdots = h_\ell = 0$ where $h_i \in \mathbb{R}[u_1, \ldots, u_m, x_1, \ldots, x_n]$ ($\mathbb{R}$ is the field of real numbers), the $u_1, \ldots, u_m$ are parameters, and the $x_1, \ldots, x_n$ are dependent variables. The conjecture has the form:

$$(\forall \mathbf{u}, \mathbf{x})[h_1 = h_2 = \cdots = h_\ell = 0 \,.\, \Rightarrow \,.\, g = 0] \tag{1}$$

where $\mathbf{u} = (u_1, \ldots, u_m)$, $\mathbf{x} = (x_1, \ldots, x_n)$ and $g = g(\mathbf{u}, \mathbf{x}) \in \mathbb{R}[\mathbf{u}, \mathbf{x}]$.

Degeneracy and Generic Truth. A theorem of the form (1) is called a *universal truth*. It turns out that the classical notion of theoremhood is more subtle, and this led Wu to formulate the notion of *generic truth*. We formalize it as follows: let $\Delta_1, \ldots, \Delta_k$ be predicates on the variables $\mathbf{u}, \mathbf{x}$. We call each Δ_i a *non-degeneracy condition*. The conjecture (1) is *generically true* relative to $\{\Delta_1, \ldots, \Delta_k\}$ if

$$(\forall \mathbf{u}, \mathbf{x})[\Delta_1, \Delta_2, \ldots, \Delta_k, h_1 = h_2 = \cdots = h_\ell = 0 \,.\, \Rightarrow \,.\, g = 0]. \tag{2}$$

Classical ruler-and-compass theorems are indeterminate in that they do not explicitly specify the degenerate conditions. Hence part of "proving a classical theorem" involves discovering a suitable set of nondegeneracy conditions. Hopefully the set is minimal is some sense (but not necessarily unique). The simplest kind of non-degeneracy condition has the form

$$\Delta : d \neq 0$$

where d is a polynomial. Call this the *first kind* of non-degeneracy condition. The *degree* of the Δ is equal to the total degree of d. If each Δ_i has degree d_i, then the *degree* of $\{\Delta_1, \ldots, \Delta_k\}$ is $\sum_{i=1}^k d_i$. Typical examples of the first kind of non-degeneracy may require two points to be distinct or two lines to be non-parallel. It is easy to see that both have degree 2.

Example 1 (continued). *The non-degeneracy conditions require the intersection points P, S and Q be not at infinity. Equivalently, the following pairs of lines are not parallel: $\{AB, DF\}$, $\{BC, FE\}$, $\{CD, EA\}$. So the degree of these non-degeneracy conditions is* 6.

Real Geometry. The *second kind* of non-degeneracy condition arises for theorems in the real field. E.g., when we define a point by the intersection of two circles, we require that these two circles

intersect. Or, when we define three collinear points A, B and C, we may require B to lie between the other two points. Such non-degeneracy conditions have the form

$$\Delta : d \geq 0$$

where d is a polynomial. It may appear that we can modify this condition using a well-known trick:

$$\Delta' : d - z^2 = 0$$

where z is a new variable. Unfortunately, it is not true that we can simply replace Δ by Δ' in the (2). This is because the new variable z is quantified by an existential quantifier, not a universal quantifier, and this is fatal for our probabilistic approach. In practice, there may be ways to circumvent this: in the Pascal example, such non-degeneracies demand that the parameters u_j $(j = 2, 3, 4, 5, 6)$ satisfy $|u_j| \leq |u_1|$. Our prover can handle nondegeneracy conditions of the second kind when put in this form. Indeed, in all the examples we looked at in [5], such a formulation is possible.

Reduction to Radical Expressions. In a ruler & compass construction, each dependent variable is a radical function of the previously introduced variables. As exemplified by Pascal's Theorem, all the dependent variables are introduced either (i) singly by a single equation (e.g., x_1 is introduced by $h_1 = 0$) or (ii) in pairs by two equations (e.g., x_6, x_7 are introduced by $h_6 = h_7 = 0$). As all equations are at most quadratic, the x_i's can be replaced by radical expressions involving the u_j's. Let $G = G(\mathbf{u})$ be the radical expression after such a substitution into $g(\mathbf{u}, \mathbf{x})$. The universal truth conjecture (1) now says

$$(\forall \mathbf{u})[G = 0],$$

with an analogous statement for generic truth. Another issue arises: each radical is determined only up to a $\pm$ sign. Hence, if there are r radicals in G, we must replace $G = 0$ by the system of 2^r radical expressions, $G_1 = G_2 = \cdots = G_{2^r} = 0$, in which each of the 2^r possible sign combinations are used. If a single function $G^*(\mathbf{u})$ is desired, we can use $G^* = \sum_{i=1}^{2^r} G_i^2$. The appearance of "$2^r$" in this expression may be disturbing from a complexity viewpoint. Several observations suggest that this is not serious in practice. First, r is typically small ($r = 5$ in Pascal's theorem). Next, we can reduce the number of summands in G^* from the worst case of 2^r terms. There are two ways this can happen: (A) Symmetries in the problem may arise so that many of the G_i's can be omitted. (B) Certain sign combinations may be excluded by the nature of the construction and/or theorem so that G^* may represent a sum of less than 2^r radical expressions. In particular, using (A) and (B), we can always omit half of the summands in standard geometric theorems. Thus, 2^{r-1} terms suffice in G^*.

Example 2 (Butterfly Theorem) We illustrate the reduction in the number of terms in G^* using the Butterfly Theorem in [5, Example 2.4, p. 9]. The theorem concerns 4 co-circular points A, B, C and D. Let O be the center of this circle and E be the intersection of AC and BD. The points A, B, C, D, E form a "butterfly". If the line perpendicular to OE and passing through E intersects the lines AC and BD at E and F (respectively), then the theorem says that segments EF and EG have the same length. There are 3 quadratic equations in formulating this theorem (so $r = 3$). In the construction described by Chou, the point E is placed at the origin $(0, 0)$ and O is placed at $(u_1, 0)$. A is freely placed at (u_2, u_3). The point C is now completely determined, and has two possible solutions. In one solution, C and A coincide, and the nature of the theorem excludes this case. This illustrates (B) above. Next, the points B is freely chosen on the circle (and this introduces one parameter). Again there are two possible solutions. But it is clear by symmetry that we can arbitrarily choose one of them without loss of generality. Therefore, G^* only needs two terms (corresponding to choosing the 2 solutions for D). ■

3 Randomized Zero Testing for Radical Expressions

3.1 Straight Line Programs

We need to generalize expressions to *straight line programs* (SLP). A SLP π is a sequence of *steps* where each step is an assignment to a new *programming variable*. The ith step of a SLP has one of the forms

$$z_i \leftarrow x_i \circ y_i, \qquad (\circ \in \{+, -, \times, \div\}) \tag{3}$$

$$z_i \leftarrow \sqrt{x_i} \tag{4}$$

where z_i is a newly introduced programming variable, x_i and y_i are either real constants, *input variables* or programming variables introduced in some earlier steps. Alternatively, we call an input variable an *independent variable* (or, *parameter*) and a programming variable a *dependent variable*. These x_i and y_i are said to be *used* in the ith step. The last introduced variable is called the *main variable* and it is never used. In general, a SLP can have branching steps. But this possibility is not considered in this paper.

An *expression* is a SLP where, with the exception of the main variable, each programming variable is used exactly once. Underlying each SLP is a labeled and ordered *dag* (directed acyclic graph) defined in the obvious way: each node corresponds to a constant or variable in the SLP. We often use the terms "nodes" and "variables" interchangeably. For the steps in (4) (resp., (3)), we introduce edges that are directed from x_i (resp., x_i and y_i) to z_i. We use standard graph-theoretic terminology to talk about this dag: sinks, sources, predecessor/successor nodes, etc. If (u, v) is an edge of the dag, we call u the *predecessor* of v, and call v the *successor* of u. The nodes labeled by input variables or constants are *source nodes* while the non-source are labeled by programming variables. The sources may be called *leaves* in case the dag is a tree. The non-source nodes are associated with an operation $(\pm, \times, \div, \sqrt{\cdot})$ – so we may speak of "radical nodes", "multiplication nodes", etc. Variables that are not used correspond to *sink nodes* in the dag. The main variable corresponds to a sink node which we call *root*. The *radical depth* of a node u is the maximum number of radical nodes in a path from u to any root node, inclusive of the end points. Thus, if u is a radical node, then the radical depth of u is at least 1. For each node u, its *induced dag* is the subdag comprising all the nodes that can reach u along a path. A SLP is said to be *rooted* if the root is the unique sink. The dags corresponding to expressions are ordered trees (hence rooted). Our SLP's are assumed rooted unless otherwise noted.

Values. Let $\mathbf{u} = (u_1, \ldots, u_m)$ be the input variables. For each variable u in a SLP π, we inductively define its *value* to be an appropriate element $val_\pi(u)$ in an algebraic extension of $\mathbb{Q}(\mathbf{u})$. The extension is obtained by adjunction of square roots. The *value* of π is the value of its main variable. More precisely, let $Q_0 = \mathbb{Q}(\mathbf{u})$ and define the tower of extensions defined by π to be

$$Q_0 \subseteq Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_r \tag{5}$$

where $Q_i := Q_{i-1}(\sqrt{\alpha_i})$ and the ith square-root in π has operand $\alpha_i \in Q_{i-1}$. A SLP π is also said to *compute* a collection $V \subseteq Q_r$ of values provided each $v \in V$ is the value of some variable in π.

Rational Degrees. Let x be a node in a SLP π. We define the *rational degree* $\mathrm{rdeg}_\pi(x)$ of x (the subscript π is usually dropped). We need some auxiliary notions. For any node or variable x, let $\mathrm{RAD}(x)$ denote the set of radical nodes in the subdag of π rooted at x. Write $\mathrm{RAD}(x, y)$ for $\mathrm{RAD}(x) \setminus \mathrm{RAD}(y)$ (set difference). Also let $\rho(x) := |\mathrm{RAD}(x)|$ and $\rho(x, y) := |\mathrm{RAD}(x, y)|$. We will inductively define $\mathrm{rdeg}(x)$ to be a pair of natural numbers $(a, b) \in \mathbb{N}^2$, but usually write it as "$a : b$". These two numbers are the "upper" and "lower" degrees of x and denoted $\mathrm{udeg}(x)$ and $\mathrm{ldeg}(x)$. Thus,

$$\mathrm{rdeg}(x) = \mathrm{udeg}(x) : \mathrm{ldeg}(x).$$

Assuming $\mathrm{rdeg}(x) = a_x : b_x$ and $\mathrm{rdeg}(y) = a_y : b_y$, we inductively define $\mathrm{rdeg}(z)$ using the table:

z	$\mathrm{udeg}(z)$	$\mathrm{ldeg}(z)$
constant	0	0
parameter	1	0
$x \times y$	$a_x 2^{\rho(y,x)} + a_y 2^{\rho(x,y)}$	$b_x 2^{\rho(y,x)} + b_y 2^{\rho(x,y)}$
$x \div y$	$a_x 2^{\rho(y,x)} + b_y 2^{\rho(x,y)}$	$b_x 2^{\rho(y,x)} + a_y 2^{\rho(x,y)}$
$x \pm y$	$\max(a_x 2^{\rho(y,x)} + b_y 2^{\rho(x,y)}, b_x 2^{\rho(y,x)} + a_y 2^{\rho(x,y)})$	$b_x 2^{\rho(y,x)} + b_y 2^{\rho(x,y)}$
$\sqrt{x}$	a_x	b_x

The *rational degree* of the SLP π is defined to be $a : b$ where $a = \max_x \mathrm{udeg}(x)$, $b = \max_x \mathrm{ldeg}(x)$, and x ranges over the nodes in π. Note that if π is division-free, then $\mathrm{ldeg}(x) = 0$ for all x.

Alternative Approach. It is useful to have an alternative approach to rdeg which does not involve $\rho(x, y)$ or $\rho(y, x)$. In particular, we define $\mathrm{rdeg}_2(z) = \mathrm{udeg}_2(z) : \mathrm{ldeg}_2(z)$ inductively using the following table: as before, we assume $\mathrm{rdeg}_2(x) = a_x : b_x$ and $\mathrm{rdeg}_2(y) = a_y : b_y$.

z	$\mathrm{udeg}_2(z)$	$\mathrm{ldeg}_2(z)$
constant	0	0
parameter	1	0
$x \times y$	$a_x + a_y$	$b_x + b_x$
$x \div y$	$a_x + b_y$	$b_x + a_y$
$x \pm y$	$\max\{a_x + b_y, b_x + a_y\}$	$b_x + b_y$
$\sqrt{x}$	$\frac{a_x}{2}$	$\frac{b_x}{2}$

Notice that these degrees are no longer natural numbers but binary fractions. The following lemma gives the connection between the two definitions of rdeg.

Lemma 1 *For any variable z in a SLP, we have*

$$\mathrm{udeg}(z) = 2^{\rho(z)}\,\mathrm{udeg}_2(z), \qquad \mathrm{ldeg}(z) = 2^{\rho(z)}\,\mathrm{ldeg}_2(z).$$

3.2 Equivalent Transformations

Two variables (resp. SLP's) are said to be *equivalent* if they have the same value. Transformations of an SLP that do not change its value are called *equivalent transformations* (but the set of computed values may change). Equivalent transformations may change the rational degree, as when applying the distributive law:

$$z(x + y) \Rightarrow zx + zy. \tag{6}$$

It is easy to verify that the rational degree of the left-hand side is at most that of the right-hand side. We next show that the rational degree is preserved in the absence of division (but allowing radicals):

Lemma 2 *If π is division-free, then the transformation (6) preserves* rdeg *of π. In particular,*

$$\mathrm{rdeg}(z(x + y)) = \mathrm{rdeg}(zx + zy).$$

The proof requires various identities such as $\mathrm{RAD}(xy, z) = \mathrm{RAD}(x, z) \uplus \mathrm{RAD}(zy, zx)$ where $\uplus$ denotes disjoint union. Next, we show that applying the associative laws for multiplication and addition does not affect rational degree. This follows from the following general result:

Lemma 3 *Let x_i be variables in π and $r_i = |\operatorname{RAD}(x_1, \ldots, x_k) \setminus \operatorname{RAD}(x_i)|$. Then*

$$\operatorname{rdeg}(\prod_{i=1}^{k} x_i) = \sum_{i=1}^{k} \operatorname{rdeg}(x_i) 2^{r_i} \qquad \operatorname{rdeg}(\sum_{i=1}^{k} x_i) = \max_{i=1}^{k} \operatorname{rdeg}(x_i) 2^{r_i}$$

The above lemma justifies a generalization of SLP's in which we allow addition nodes and multiplication nodes to take an arbitrary number of arguments. These are called "sum" or $\sum$-nodes, and "product" or $\prod$-nodes, respectively. Such an SLP is called a *generalized SLP*. A path in a generalized SLP dag is said to be *alternating* if along the path, no two consecutive nodes are $\sum$-nodes and no two consecutive nodes are $\prod$-nodes. The SLP is *alternating* if every path is alternating. Clearly, any SLP can be made alternating without changing its rational degree.

3.3 Preparation

A SLP in which the last three steps has the form

$$\begin{aligned} &\cdots \\ &x \leftarrow \sqrt{w_C} \\ &y \leftarrow x \times w_B \\ &z \leftarrow y + w_A \end{aligned}$$

is said to be *prepared* (or in prepared form). Here w_A, w_B, w_C are variables or constants. Thus z is the main variable, and x is the last radical variable to be introduced. Intuitively, the radical x has been brought up as close to the root as possible, in preparation for a transformation (to be introduced) to remove the radical. We also call x the *prepared variable*. If the values of w_A, w_B, w_C are given by the expressions A, B, C (resp.) then the value of z is given by the expression

$$A + B\sqrt{C}.$$

Note special forms of this expression when $A = 0$ or $B = 1$, or both. If the SLP has no square roots, it is considered prepared already. Our goal is to prepare a given SLP, and to bound the resulting rational degree.

Let us now prepare a radical node A_0 with radical depth 1. Assume the SLP is division-free. Let A_n, B_n be expressions ($n \geq 0$). The expressions E_n, for $n \geq 0$ is defined inductively as follows: $E_0 = A_0 \times B_0$, and for $n \geq 1$,

$$E_n = (E_{n-1} + A_n)B_n = ((E_{n-2} + A_{n-1})B_{n-1} + A_n)B_n = \cdots.$$

To show the dependence of E_n on the A_n's and B_n's, we may also write $E_n = E_n(A_0, B_0, A_1, B_1, \ldots, A_n, B_n)$. Viewed as a tree, E_n is essentially a single alternating path from the root down to A_0. This path is left-branching only and the root is a $\times$-node. Also write: $B_{(n)} := \prod_{j=0}^{n} B_j$.

Lemma 4 *For $n \geq 1$, the expression $E_n(A_0, B_0, \ldots, A_n, B_n)$ is equivalent to the expression*

$$E'_n := (A_0 \times B_{(n)}) + E_{n-1}(A_1, B_1, \ldots, A_n, B_n)$$

Moreover, if E_n is division-free, then $\operatorname{rdeg}(E_n) = \operatorname{rdeg}(E'_n)$.

This is illustrated in the case $n = 2$ by figure 1. Note that the variable A_0 is prepared in E'_n. Actually, E_n in this lemma can be a generalized SLP so that the A_i, B_i's need not be distinct and

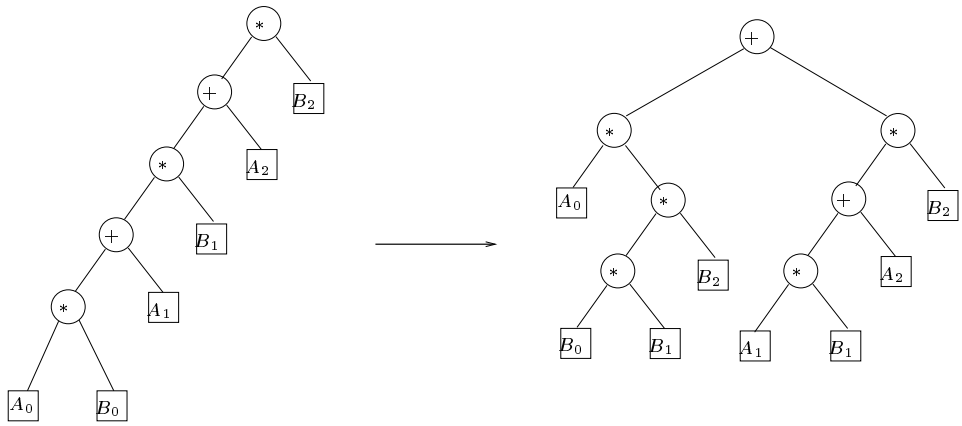


Figure 1: The transformation $E_2 \mapsto E_2'$.

the nodes can be $\sum$- and $\prod$-nodes. Then there is a corresponding equivalent SLP E_n'; this is the version that we will use in the next theorem.

We address the problem of multiple uses of a node. A node u is *used* k times if there are k distinct paths from the root to u. If a radical node u of radical depth 1 is used k times, then if we judiciously apply the previous lemma k times, each time eliminating one "use" of u, we obtain:

Theorem 5 *Suppose π is a division-free SLP and u is a radical node in π with radical depth of 1. Then we can transform π into an equivalent SLP π' such that* $\mathrm{udeg}(\pi) \geq \mathrm{udeg}(\pi')$. *Moreover, either no node in π' has the value $val_\pi(u)$ or else, there is a node u' in π' with the following properties:*

1. *u' is the prepared variable in π'*

2. *u' is the unique node in π' with value $val_\pi(u)$.*

Proof. (Sketch) We may assume that π is a generalized, alternating SLP. Fix any path p from u to the root and we may assume that this alternating sum-product path has the same form as the path from A_0 to the root of E_n. We then apply the previous lemma in which u now plays the role of the node A_0 in E_n. This collapses the path p to length 2, as in the lemma. We can repeat this process for another path p' (if any) from u to the root. Again, the collapsed p' can be assumed to have length 2. Now, we can merge these two paths by applying the reverse distributive law (6). This application of the law does not change the rational degree of u. Moreover, the number of paths from u to the root has decreased by 1. By repeating this process, we finally ensure that there is a unique path from u to the root, and the depth of u is 2. At this point, the SLP has the desired form: u is prepared. It is also clear that if there are other nodes with the same value as u, they can also be merged with u by the same process. Hence, u will be the unique node with value $val_\pi(u)$. **Q.E.D.**

We say that π' is obtained by the process of "preparing" u in π. Note that this argument also works if all paths from u to the root are division free. In this case the transformation may actually lower the rational degree when the reverse distributive law is applied.

3.4 Main Result

Let π be a SLP whose value is $V = V(\mathbf{u}) \in \mathbb{Q}_r$ (see (5)). We define the real function $f_\pi : \mathbb{R}^m \to \mathbb{R}$ where $f_\pi(a_1, \ldots, a_m)$ is the value of the main variable in π when we evaluate each dependent variable

at $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^m$, following π in a step-by-step fashion. The *domain* of f_π comprises those $\mathbf{a} \in \mathbb{R}^m$ where $f_\pi(\mathbf{a})$ is defined. Similarly, we define an associated real function $f_V : \mathbb{R}^m \to \mathbb{R}$. Note that the domain of f_π is always a subset of f_V. The following example shows that it may be a proper subset: let π compute the value $V = \sum_{i=0}^{n-1} x^i$ using Horner's rule, and let π' compute the same V using the formula $V = \frac{x^n - 1}{x - 1}$. Then π and π' are equivalent, but $\pi(1) = n$ while $\pi'(1)$ is undefined. The domain of π (and V) is $\mathbb{R}$ but the domain of π' is $\mathbb{R} - \{1\}$.

Theorem 6 *Suppose $V = V(\mathbf{u})$ is the non-zero value of a rooted division-free SLP π. Then there exists a non-zero polynomial $P(\mathbf{u})$ such that* $\mathtt{Zero}(V) \subseteq \mathtt{Zero}(P)$ *with* $\deg P(\mathbf{u}) \leq \mathrm{udeg}(\pi)$.

Proof. We show the existence of the polynomial $P(\mathbf{u})$ by induction on the number r of square roots in π. For $r = 0$, the result holds because V is already a polynomial of degree $\mathrm{udeg}(\pi)$.

Assume $r > 0$ and let u be a radical node of radical depth 1 in π. We prepare u, leading to an equivalent SLP (which we still call π). The udeg of π is unchanged by this transformation. If C is the value of u, then the value of π can be written as

$$V = A + B\sqrt{C}$$

where A, B, C belongs to Q_{r-1} (recall that values of programming variable introduced before the rth root extraction belongs to the field Q_{r-1}, by definition of Q_{r-1}). If $B = 0$ then $V = A$ and the result is true by the inductive hypothesis applied to A (which has $\leq r - 1$ square roots). Otherwise, by applying some further (obvious) transformations, we transform π to some π' whose value is

$$V' = A^2 - B^2 C. \tag{7}$$

Note that π' has $\leq r - 1$ square-roots. If $V' = 0$ then $0 = V' = (A + B\sqrt{C})(A - B\sqrt{C})$. Since Q_r is a UFD and $V = A + B\sqrt{C} \neq 0$ (by assumption), we conclude that $A - B\sqrt{C} = 0$, i.e., $\sqrt{C} = A/B \in Q_{r-1}$. Thus $V = A + B\sqrt{C} = 2A$. Then V can be computed by some SLP with $\leq r - 1$ square-roots, and the result follows by inductive hypothesis.

So assume $V' \neq 0$. By induction, $\mathtt{Zero}(V') = \mathtt{Zero}(A^2 - B^2 C) \subseteq \mathtt{Zero}(P)$ for some P with $\deg(P) \leq \mathrm{udeg}(V')$. Since $\mathtt{Zero}(V) \subseteq \mathtt{Zero}(V')$, it remains to show that $\mathrm{udeg}(V') \leq \mathrm{udeg}(V)$. We have

$$\begin{aligned}
\mathrm{udeg}(V) &= \mathrm{udeg}(A + B\sqrt{C}) \\
&= \max\{\mathrm{udeg}(A) 2^{\rho(B\sqrt{C}, A)}, \mathrm{udeg}(B\sqrt{C}) 2^{\rho(A, B\sqrt{C})}\} \\
&\geq \max\{\mathrm{udeg}(A) 2^{1+\rho(B^2 C, A)}, \left[\mathrm{udeg}(B) 2^{\rho(\sqrt{C}, B)} + \mathrm{udeg}(C) 2^{\rho(B, \sqrt{C})}\right] 2^{\rho(A, B^2 C)}\} \\
&= \max\{2\,\mathrm{udeg}(A) 2^{\rho(B^2 C, A)}, \left[(\mathrm{udeg}(B^2)/2) 2^{1+\rho(C, B^2)} + \mathrm{udeg}(C) 2^{\rho(B^2, C)}\right] 2^{\rho(A, B^2 C)}\} \\
&\geq \max\{\mathrm{udeg}(A^2) 2^{\rho(B^2 C, A)}, \left[\mathrm{udeg}(B^2) 2^{\rho(C, B^2)} + \mathrm{udeg}(C) 2^{\rho(B^2, C)}\right] 2^{\rho(A^2, B^2 C)}\} \\
&= \mathrm{udeg}(A^2 - B^2 C) = \mathrm{udeg}(V').
\end{aligned}$$

Q.E.D.

3.5 Presence of Division

What if the SLP is not division-free? Note that the presence of division is very common. For instance, when we intersect two lines in the construction, it gives rise to an expression with division.

There is a well-known transformation to move all divisions towards the root, merging them as we go. An instance of this transformation is

$$\frac{A}{B} + \frac{A'}{B'} \Rightarrow \frac{AB' + A'B}{BB'}.$$

Unfortunately, the number of radical nodes may be doubled because if we move a division node past a radical node, we obtain two radical nodes:

$$\sqrt{\frac{A}{B}} \Rightarrow \frac{\sqrt{A}}{\sqrt{B}}.$$

Hence we give two versions of this transformation in the following lemma: in version (i) we do not move any division node past a radical node, and in version (ii) we remove all but at most one division node.

Lemma 7 (Elimination of Division) *Let π be a rooted SLP.*
(i) There is an equivalent SLP π' in which each division node is either the root of π or the child of a radical node. Moreover, $\mathrm{rdeg}(\pi') = \mathrm{rdeg}(\pi)$ *and π' has the same number of radical nodes as π.*
(ii) There is an equivalent SLP π'' with only one division node which is also the root. In this case $\mathrm{rdeg}(\pi'') \leq 2^r \mathrm{rdeg}(\pi)$.

The proof of (ii) exploits the alternative definition of $\mathrm{udeg}(u)$. The value of the SLP π'' has the form A/B where A, B are division-free. Intuitively, to check if $A/B = 0$, we check if $A = 0$ subject to $B \neq 0$. Since A is division-free, we may apply main theorem (see next Section). This effectively amounts to doubling the number of square roots to prove a theorem involving division.

4 Proving by Random Examples

We show how to use our main result to prove theorems about ruler & compass constructions. According to Section 2, this amounts to verifying if a radical expression $G^*(\mathbf{u})$ is identically zero (subject to non-degeneracy conditions). Let $\pi(\mathbf{u})$ be the natural SLP which computes the values of all the dependent variables in a ruler & compass construction, and whose value is the polynomial thesis $G^*(\mathbf{u})$. We give a crude upper estimate on the rdeg of each node in π, assuming that each "stage" of our construction introduces one or two variables, and a corresponding number of equations. Assume an incremental construction with m parameters, n dependent variables, k stages, and r quadratic equations ($r \leq 2k$ and $k \leq n$).

Lemma 8 *If the dependent variable x is introduced in the ith stage, then* $\mathrm{rdeg}_\pi(x) \leq 8^i$, *i.e.,* $\mathrm{udeg}_\pi(x) \leq 8^i$, $\mathrm{ldeg}_\pi(x) \leq 8^i$.

Corollary 9 *Let the thesis polynomial be $g(\mathbf{u}, \mathbf{x})$ with* $\deg(g) = d$, *and $G(\mathbf{u})$ be any of the 2^r radical expressions derived from $g(\mathbf{u}, \mathbf{x})$ by eliminating dependent variables. Then* $\mathrm{rdeg}_\pi(G) \leq td8^k$ *where $g(\mathbf{u}, \mathbf{x})$ has t terms.*

Note that t is at most $\binom{m+n+d}{d}$. Moreover, $d \leq 2$ in most classical geometric theorems. In our implementation, instead of relying on this crude upper bound, we actually compute the actual bounds on rdeg to achieve better performance. By applying lemma 7(ii) to π, we obtain π'' with one division at the root, and $\mathrm{rdeg}(\pi'') \leq 2^r \mathrm{rdeg}(\pi)$. Now the value of π'' (which is G^*) has the form A/B where A, B are division-free. Moreover, $\mathrm{rdeg}_{\pi''}(G^*) \leq td2^{2r+3k}$. Clearly, $\mathtt{Zero}(A/B) \subseteq \mathtt{Zero}(A)$. Without loss of generality, assume $A \neq 0$. By our main theorem, $\mathtt{Zero}(A) \subseteq \mathtt{Zero}(P)$ for some polynomial P of degree $\leq td2^{2r+3k}$. Then we invoke a simple form of Schwartz's lemma:

Fact 1 *Let $P(\mathbf{u})$ be a non-zero polynomial of degree at most D. If each a_i ($i = 1, \ldots, m$) is randomly chosen from a finite set $S \subseteq \mathbb{R}$. Then the probability that $P(a_1, \ldots, a_m) = 0$ is at most $D/|S|$.*

If we randomly pick the values $\mathbf{a} = (a_1, \ldots, a_m) \in S^m$, and $|S| = d2^{c+2r+3k}$ (for any $c \geq 1$) then the "error probability" of our procedure is given by $\mathtt{Pr}\{A(\mathbf{a}) = 0\} \leq \mathtt{Pr}\{P(\mathbf{a}) = 0\} \leq 2^{-c}$. This constitutes our probabilistic verification of the universal truth of "$G^*(\mathbf{u}) = 0$".

An alternative to testing $G^*(\mathbf{u}) = 0$ is view the problem as testing the simultaneous vanishing of a set of polynomial $\mathcal{G} := \{G_1(\mathbf{u}), \ldots, G_{2^r}(\mathbf{u})\}$. This reduces the complexity in two ways:

- The root bound (which determines the precision necessary to numerically determine the sign of radical expressions in the Core Library) is smaller.
- The size of the test set S is smaller.

We also have a further choice when testing $\mathcal{G}$: we can randomly choose some G_i to test for its vanishing, or we can choose to randomly test all the G_i's for their vanishing. However, the random choice of G_i may not be the most efficient way to test a theorem. See the full paper for details.

Degeneracies of the First Kind. We next address the generic truth of "$G^*(\mathbf{u}) = 0$". The notion of "error probability" becomes an interesting issue. First consider only non-degeneracy conditions of the first kind, $\Delta : \delta \neq 0$. For simplicity, assume the ith ruler & compass construction step introduces exactly one such condition, $\delta_i \neq 0$, of degree ≤ 2. Since there are k stages of construction, the non-degeneracy condition becomes $\delta^* := \delta_1 \delta_2 \cdots \delta_k \neq 0$. The degree of δ^* is thus at most $2k$.

There are two natural models of what it means to have an "error probability" $\leq 2^{-c}$: (A) The "strict model" says that our sample space is now restricted to $S^m \setminus \{\mathbf{a} : \delta(\mathbf{a}) = 0\}$. (B) Alternatively, we can say that the sample space is still S^m but the theorem is trivially true at $S^m \cap \{\mathbf{a} : \delta(\mathbf{a}) = 0\}$. This amounts to adding the polynomial δ^* into the set $\mathcal{G}$. Either probability model may be used in the next theorem:

Theorem 10 *Conjectures about ruler & compass constructions with non-degeneracies of the first kind can be verified with error probability $\leq 2^{-c}$ in time polynomial in the parameters r, k, c and $\log d$.*

Degeneracies of the Second Kind. As noted, degeneracies of the second kind can often be reduced to simple constraints on the domains of the parameters, possibly depending on the values of other parameters. For instance, we noted that in Pascal's Theorem, the parameters u_i ($i = 2, \ldots, 6$) must satisfy $|u_i| \leq |u_1|$. Our prover can handle such degeneracies by exploiting the following more general form of fact 1: define the *generalized degree* of $p(x_1, \ldots, x_n)$ to be $(d_1, \ldots, d_n)$ where the degree of p is d_1 when viewed as a polynomial in x_1 and its leading coefficient inductively has generalized degree $(d_2, \ldots, d_n)$. Suppose $S_1, \ldots, S_n$ are finite sets of real numbers, then it can be shown that if we choose $(u_1, \ldots, u_n)$ randomly from $S_1 \times S_2 \times \cdots S_n$, the probability that p is non-zero and $p(u_1, \ldots, u_n) = 0$ is at most

$$\frac{d_1}{|S_1|} + \cdots + \frac{d_n}{|S_n|}.$$

The main extra complexity caused by this version of our prover is that we need to evaluate the parameters at rational values (instead of just at integer values).

Our current implementation does not handle the second kind of degeneracy in the above way[1]. Instead, it detects when an example $\mathbf{a} \in S^m$ is degenerate, discards it and generates another example,

[1] We plan to rectify this in the future.

etc. Under probability model (A) above, this means that we do not have an á priori bound on the running time, but the error probability is correct. Of course, under model (B), there is no need to generate another example; but this does not seem like a reasonable model.

Timing. We prove some theorems from Chou [5]. The timings are for two values of c (this means the probability of error is at most 2^{-c}). We also arbitrarily "perturb" the hypothesis of each theorem by randomly changing one coefficient of one of the input polynomials, and report their timings as well. These are all false theorems, naturally. Our tests were performed on a Sun UltraSPARC-IIi (440 MHz, 512 MB). The times are all in seconds, and represent the average of 6 runs each. The prover uses Core Library, Version 1.3. Actually, the library is directly modified so that we compute the exact rational degrees of the expressions (rather than use the estimates of the lemma 8). For comparison, we include the timings reported by Chou [5] using the approaches of Wu and of Gröbner bases. The final column in the table gives the page number in Chou's book [5].

No.	Theorem	$c = 10$	$c = 20$	Perturbed	Char Set	Gröbner	Page
1	Pappus	0.020	0.020	0.007	1.52	33.32	100
2	Pappus Point	0.152	0.147	0.025	4.87	67.62	100
3	Pappus-dual	0.017	0.023	0.008	1.45	25.53	111
4	Nehring	4.380	5.870	0.102	4.15	159.3	115
5	Chou-46	0.059	0.083	0.022	88.13	37.65	124
6	Ceva	0.027	0.033	0.010	1.12	3.47	264
7	Simson	70.318	39.000	0.017	1.22	5.02	240
8	Pascal	1715.8	2991.6	0.037	29.6	>14400	103

Let r be the number of square roots in the radical expression representing a theorem. If $r = 0$, we say the theorem is linear. A large part[2] of the 512 theorems in Chou's book are linear. Only the last two theorems (Simson and Pascal) in the above list are non-linear, with $r = 1$ and $r = 5$, respectively. Evidently non-linear theorems represent a challenge for our current system. Another comment about the timing is that we randomly test only one of the 2^r possible thesis radical expressions. So, in the "worst case", the actual timing should be more than our reported time by a factor of 2^{r-1}. Thus only our timing for Pascal is affected.

It is interesting to note that we have never observed a single wrong conclusion from our probabilistic tests – all the theorems are reported as true, and all perturbed theorems are reported as false. In some sense, that is not surprising because the probabilistic bounds based on Schwartz's lemma seem overly conservative in all real situations.

The running times for linear theorems are pretty consistent across different runs. However, for the non-linear theorems, the timing can show much more variation. This is not unexpected since the running time depends on the bit size of the random example. A more prominent behavior comes from the clustering of times around certain values. For instance, for Simson ($c = 20$), the times cluster around 10 seconds and around 70 seconds. This "multimodal" behavior of the timings are again seen in Pascal. This can be attributed to the random choice of signs for the radicals in nonlinear theorems. This may also account for the curious relative times for Simson $c = 10$ and $c = 20$.

The performance of our library is critically dependent of good root bounds (an area of research that we are actively working on [15]). It should be possible to exploit prover-specific techniques to improve the speed, but this has not been done. There are several issues to bear in mind when comparing our method with Wu's method:

[2]The theorems in Chou's book include an original list of 366 theorems from [4], of which 219 are reported to be linear [5, p. 12].

- Chou's timings would look considerably better using hardware available today.
- The actual theorems proved by Wu's method is not strictly comparable to ours in two important aspects: Wu's method proves theorems about complex geometry while ours is about real geometry (the latter seems more useful in applications). On the other hand, Chou's algorithm is deterministic while ours is probabilistic.
- Our method is extremely effective for discarding wrong or perturbed conjectures. It is unclear if Wu's method will be much faster for perturbed theorems, since the algorithm would still have to execute the same basic steps. The ability to quickly reject false theorems is extremely useful in applications where the user has many conjectures to check but most of the conjectures are likely to be false.
- One of the strengths of Wu's methods (as compared to Gröbner bases, say) is its ability to discover non-degeneracy conditions. A similar capability is embedded in our approach – this simply amounts to detecting when a construction step is ill-defined.

5 Final Remarks

In this paper, we have developed a generalization of the Schwartz-Zippel randomized zero test for the class of radical expressions. Such a test is expected to have many applications as radical expressions are quite common. Here, we focus on their use in proving theorems about ruler & compass constructions. Some features of our prover are:

- It proves theorems about real (rather than complex) geometry.
- It is probabilistic, so that speed can be traded-off against error probability.
- It detects wrong conjectures very quickly.
- It is extremely effective for linear theorems (the majority of the theorems in [5]).
- It exploits the special nature of ruler & compass constructions.

Because of the last feature, our approach may ultimately prove to be more efficient for *this* class of problems than other more general techniques. However, our results so far have not been conclusive in the case of non-linear theorems. The following are some open problems:

- Improve our zero test for straight line programs that involve division. Our current bound on the udeg amounts to doubling the number of radical nodes, a serious penalty.
- Extend our probabilistic approach to handle non-degeneracies of the second kind ($d > 0$).
- Develop techniques to make our approach faster for nonlinear theorems.

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