Some Recent Tools and a BDDC Algorithm for 3D Problems in H(curl)

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Summary. We present some recent domain decomposition tools and a BDDC algorithm for 3D problems in the space $H(\mathrm{curl};\Omega)$. Of primary interest is a face decomposition lemma which allows us to obtain improved estimates for a BDDC algorithm under less restrictive assumptions than have appeared previously in the literature. Numerical results are also presented to confirm the theory and to provide additional insights.

1 Introduction

We investigate a BDDC algorithm for three-dimensional (3D) problems in the space $H_0(\mathrm{curl};\Omega)$. The subject problem is to obtain edge finite element approximations of the variational problem: Find $\boldsymbol{u} \in H_0(\mathrm{curl};\Omega)$ such that

$$a_\Omega(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v})_\Omega \quad \forall \boldsymbol{v} \in H_0(\mathrm{curl};\Omega),$$

where

$$a_\Omega(\boldsymbol{u},\boldsymbol{v}) := \int_\Omega [(\alpha \nabla \times \boldsymbol{u} \cdot \nabla \times \boldsymbol{v}) + (\beta \boldsymbol{u} \cdot \boldsymbol{v})]\, dx, \quad (\boldsymbol{f},\boldsymbol{v})_\Omega = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx.$$

The norm of $\boldsymbol{u} \in H(\mathrm{curl};\Omega)$, for a domain with diameter 1, is given by $a_\Omega(\boldsymbol{u},\boldsymbol{u})^{1/2}$ with $\alpha = 1$ and $\beta = 1$; the elements of $H_0(\mathrm{curl})$ have vanishing tangential components on $\partial\Omega$. We could equally well consider cases where this boundary condition is imposed only on one or several subdomain faces which form part of $\partial\Omega$. We will assume that $\alpha \geq 0$ and $\beta > 0$ are constant in each of the subdomains $\Omega_1,\ldots,\Omega_N$. Our results could be presented in a form which accommodates properties which are not constant or isotropic in each subdomain, but we avoid this generalization for purposes of clarity.

In the pioneering work of [11], two different cases were analyzed for FETI-DP algorithms: *Case 1:*

$$\alpha_i = \alpha \quad \text{for} \quad i = 1,\ldots,N$$

The condition number bound reported for the preconditioned operator is

$$\kappa \leq C \max_i (1 + H_i^2 \beta_i/\alpha)(1 + \log(H/h))^4, \tag{1}$$

where $H/h := \max_i H_i/h_i$.
Case 2:

$$\beta_i = \beta \quad \text{for} \quad i = 1, \ldots, N$$

for which the reported condition number bound is

$$\kappa \leq C \max_i (1 + H_i^2 \beta/\alpha_i)(1 + \log(H/h))^4. \tag{2}$$

We address the following basic questions regarding [11] in this study.

1. Is is possible to remove the assumption of $\alpha_i = \alpha$ or $\beta_i = \beta$ for all i?
2. Is it possible to remove the factor of $H_i^2 \beta_i/\alpha_i$ from the estimates?
3. Is is possible to reduce the logarithmic factor from four powers to two powers as is typical of other iterative substructuring algorithms?
4. Do FETI-DP or BDDC algorithms for 3D H(curl) problems have certain complications not present for problems with just a single parameter?

We find in the following sections that the answers are yes to all four questions. However, due to page limitations, we only consider here the relatively rich coarse space of Algorithm C of [11]. We remark that the analysis of 3D H(curl) problems with material property jumps between subdomains is quite limited in the literature. A comprehensive treatment of problems in 2D can be found in [3]. A different iterative substructuring algorithm for 3D problems is given in [6], but the authors were unable to conclude whether their condition number bound was independent of material property jumps.

2 Tools

We assume that Ω is decomposed into N non-overlapping subdomains, $\Omega_1, \ldots, \Omega_N$, each the union of elements of the triangulation of Ω. We denote by H_i the diameter of Ω_i. The interface of the domain decomposition is given by

$$\Gamma := \left(\bigcup_{i=1}^{N} \partial \Omega_i \right) \backslash \partial \Omega,$$

and the contribution to Γ from $\partial \Omega_i$ by $\Gamma_i := \partial \Omega_i \backslash \partial \Omega$. These sets are unions of subdomain faces, edges, and vertices. For simplicity, we assume that each subdomain is a shape-regular and convex tetrahedron or hexahedron with planar faces.

We assume a shape-regular triangulation $\mathscr{T}_{h_i}$ of each Ω_i with nodes matching across the interfaces. The smallest element diameter of $\mathscr{T}_{h_i}$ is denoted by h_i. Associated with the triangulation $\mathscr{T}_{h_i}$ are the two finite element spaces $W_{\text{grad}}^{h_i} \subset H(\text{grad}, \Omega_i)$ and $W_{\text{curl}}^{h_i} \subset H(\text{curl}, \Omega_i)$ based on continuous, piecewise linear, tetrahedral nodal elements and linear, tetrahedral edge (Nédeléc) elements, respectively. We could equally well develop our algorithms and theory for low order hexahedral elements.

The energy of a vector function $\boldsymbol{u} \in W_{\text{curl}}^{h_i}$ for subdomain Ω_i is defined as

$$E_i(\boldsymbol{u}) := \alpha_i (\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{u})_{\Omega_i} + \beta_i (\boldsymbol{u}, \boldsymbol{u})_{\Omega_i}, \tag{3}$$

where α_i and β_i are assumed constant in Ω_i.

Let $\boldsymbol{N}_e \in W^{h_i}_{\text{curl}}$ and $\boldsymbol{t}_e$ denote the finite element shape function and unit tangent vector, respectively, for an edge e of $\mathscr{T}_{h_i}$. We assume that $\boldsymbol{N}_e$ is scaled such that $\boldsymbol{N}_e \cdot \boldsymbol{t}_e = 1$ along e. The *edge* finite element interpolant of a sufficiently smooth vector function $\boldsymbol{u} \in H(\text{curl}, \Omega_i)$ is then defined as

$$\Pi^{h_i}(\boldsymbol{u}) := \sum_{e \in \mathscr{M}_{\bar{\Omega}_i}} u_e \boldsymbol{N}_e, \quad u_e := (1/|e|) \int_e \boldsymbol{u} \cdot \boldsymbol{t}_e \, ds, \tag{4}$$

where $\mathscr{M}_{\bar{\Omega}_i}$ is the set of edges of $\mathscr{T}_{h_i}$, and $|e|$ is the length of e. We will also make use of other sets of subdomain edges. The sets $\mathscr{M}_{\partial\Omega_i}$, $\mathscr{M}_{\mathscr{E}}$, $\mathscr{M}_{\mathscr{F}}$, and $\mathscr{M}_{\partial\mathscr{F}}$ contain the edges of $\partial\Omega_i$, subdomain edge $\mathscr{E}$, subdomain face $\mathscr{F}$, and $\partial\mathscr{F}$, respectively. We denote by $\mathscr{G}_{i\mathscr{F}}$, $\mathscr{G}_{i\mathscr{E}}$, and $\mathscr{G}_{i\mathscr{V}}$ sets of subdomain faces, subdomain edges, and subdomain vertices for Ω_i. The wire basket $\mathscr{W}_i$ is the union of all subdomain edges and vertices for Ω_i. We will also make use of the symbol $\omega_i := 1 + \log(H_i/h_i)$, and bold faced symbols refer to vector functions. We denote by $\bar{p}_i$ the mean of p_i over Ω_i.

The estimate in the next lemma can be found in several references, see e.g., Lemma 4.16 of [12].

Lemma 1. *For any $p_i \in W^{h_i}_{grad}$ and subdomain edge $\mathscr{E}$ of Ω_i,*

$$\|p_i\|^2_{L^2(\mathscr{E})} \leq C\omega_i \|p_i\|^2_{H^1(\Omega_i)}. \tag{5}$$

Lemma 2. *For any $p_i \in W^{h_i}_{grad}$, there exist $p_{i\mathscr{V}}, p_{i\mathscr{E}}, p_{i\mathscr{F}} \in W^{h_i}_{grad}$ such that*

$$p_i|_{\partial\Omega_i} = \sum_{\mathscr{V} \in \mathscr{G}_{i\mathscr{V}}} p_{i\mathscr{V}}|_{\partial\Omega_i} + \sum_{\mathscr{E} \in \mathscr{G}_{i\mathscr{E}}} p_{i\mathscr{E}}|_{\partial\Omega_i} + \sum_{\mathscr{F} \in \mathscr{G}_{i\mathscr{F}}} p_{i\mathscr{F}}|_{\partial\Omega_i}, \tag{6}$$

where the nodal values of $p_{i\mathscr{V}}$, $p_{i\mathscr{E}}$, and $p_{i\mathscr{F}}$ on $\partial\Omega_i$ may be nonzero only at the nodes of $\mathscr{V}$, $\mathscr{E}$, and $\mathscr{F}$, respectively. Further,

$$|p_{i\mathscr{V}}|^2_{H^1(\Omega_i)} \leq C\|p_i\|^2_{H^1(\Omega_i)}, \tag{7}$$

$$|p_{i\mathscr{E}}|^2_{H^1(\Omega_i)} \leq C\omega_i\|p_i\|^2_{H^1(\Omega_i)}, \tag{8}$$

$$|p_{i\mathscr{F}}|^2_{H^1(\Omega_i)} \leq C\omega_i^2\|p_i\|^2_{H^1(\Omega_i)}. \tag{9}$$

Proof. The estimates in (7-9) are standard, and follow from Corollary 4.20 and Lemma 4.24 of [12] and elementary estimates.

We note that a Poincaré inequality allows us to replace the H^1-norm of p_i by its H^1-seminorm in Lemmas 1 and 2 if $\bar{p}_i = 0$.

The next lemma is stated without proof due to page restrictions.

Lemma 3. *Let $f_i \in W^{h_i}_{grad}$ have vanishing nodal values everywhere on $\partial\Omega_i$ except on the wire basket $\mathscr{W}_i$ of Ω_i. For each subdomain face $\mathscr{F}$ of Ω_i and $Ch_i \leq d \leq H_i/C$, $C > 1$, there exists a $\boldsymbol{v}_i \in W^{h_i}_{curl}$ such that $v_{ie} = \nabla f_{ie}$ for all $e \in \mathscr{M}_{\mathscr{F}}$, $v_{ie} = 0$ for all other edges of $\partial\Omega_i$, and*

$$\|\boldsymbol{v}_i\|^2_{L^2(\Omega_i)} \leq C(\omega_i\|f_i\|^2_{L^2(\partial\mathscr{F})} + d^2\|\nabla f_i \cdot \boldsymbol{t}_{\partial\mathscr{F}}\|^2_{L^2(\partial\mathscr{F})}), \tag{10}$$

$$\|\nabla \times \boldsymbol{v}_i\|^2_{L^2(\Omega_i)} \leq C(\tau(d)\|f_i\|^2_{L^2(\partial\mathscr{F})} + \|\nabla f_i \cdot \boldsymbol{t}_{\partial\mathscr{F}}\|^2_{L^2(\partial\mathscr{F})}), \tag{11}$$

where $\boldsymbol{t}_{\partial\mathscr{F}}$ *is a unit tangent along* $\partial\mathscr{F}$*, and*

$$\tau(d) = \begin{cases} 0 & \text{if } d > H_i/C \\ d^{-2} & \text{otherwise.} \end{cases}$$

The Helmholtz-type decomposition and estimates in the next lemma will allow us to make use of and build on existing tools for scalar functions in $H^1(\Omega_i)$. We refer the reader to Lemma 5.2 of [4] for the case of convex polyhedral subdomains; this important paper was preceded by [5], which concerns other applications of the same decomposition.

Lemma 4. *For a convex and polyhedral subdomain* Ω_i *and any* $\boldsymbol{u}_i \in W^{h_i}_{curl}$*, there is a* $\boldsymbol{q}_i \in W^{h_i}_{curl}$*,* $\boldsymbol{\Psi}_i \in (W^{h_i}_{grad})^3$*, and* $p_i \in W^{h_i}_{grad}$ *such that*

$$\boldsymbol{u}_i = \boldsymbol{q}_i + \Pi^{h_i}(\boldsymbol{\Psi}_i) + \nabla p_i, \tag{12}$$

$$\|\nabla p_i\|_{L^2(\Omega_i)} \leq C\|\boldsymbol{u}_i\|_{L^2(\Omega_i)}, \tag{13}$$

$$\|\boldsymbol{\Psi}_i\|_{L^2(\Omega_i)} \leq C\|\boldsymbol{u}_i\|_{L^2(\Omega_i)}, \tag{14}$$

$$\|h_i^{-1}\boldsymbol{q}_i\|^2_{L^2(\Omega_i)} + \|\boldsymbol{\Psi}_i\|^2_{H^1(\Omega_i)} \leq C\|\nabla\times\boldsymbol{u}_i\|^2_{L^2(\Omega_i)}. \tag{15}$$

Lemma 5. *For any* $\boldsymbol{u}_i \in W^{h_i}_{curl}$ *with* $u_{ie} = 0$ *for all* $e \in \mathscr{M}_{\mathscr{E}}$ *and* $\mathscr{E} \in \mathscr{G}_{i\mathscr{E}}$*, there exists a* $\boldsymbol{v}_i \in W^{h_i}_{curl}$ *such that* $v_{ie} = u_{ie}$ *for all* $e \in \mathscr{M}_{\partial\Omega_i}$*,*

$$\boldsymbol{v}_i = \sum_{\mathscr{F}\in\mathscr{G}_{i\mathscr{F}}} \boldsymbol{v}_{i\mathscr{F}}, \tag{16}$$

where $v_{i\mathscr{F}e} = 0\ \forall e \in \mathscr{M}_{\partial\Omega_i} \setminus \mathscr{M}_{\mathscr{F}}$*. Further,*

$$E_i(\boldsymbol{v}_{i\mathscr{F}}) \leq C\omega_i^2 E_i(\boldsymbol{u}_i), \tag{17}$$

where the energy E_i *is defined in (3).*

Proof. Let p_i in (12) be chosen so $\bar{p}_i = 0$. This is possible since a constant can be added to p_i without changing its gradient. Because $u_{ie} = 0$ for all $e \in \mathscr{M}_{\mathscr{E}}$, it follows from Lemmas 1 and 4 and elementary estimates that

$$\begin{aligned}\|\nabla p_i \cdot \boldsymbol{t}_{\mathscr{E}}\|^2_{L^2(\mathscr{E})} &\leq \|(\Pi^{h_i}(\boldsymbol{\Psi}_i) + \boldsymbol{q}_i)\cdot\boldsymbol{t}_{\mathscr{E}}\|^2_{L^2(\mathscr{E})} \\ &\leq C\omega_i\|\nabla\times\boldsymbol{u}_i\|^2_{L^2(\Omega_i)}.\end{aligned} \tag{18}$$

For each subdomain face $\mathscr{F}$ of Ω_i, we find from Lemmas 2 and 4 that

$$\|\nabla p_{i\mathscr{F}}\|^2_{L^2(\Omega_i)} \leq C\omega_i^2\|\boldsymbol{u}_i\|^2_{L^2(\Omega_i)}. \tag{19}$$

Define

$$p_{i\mathscr{W}} := \sum_{\mathscr{V}\in\mathscr{G}_{i\mathscr{V}}} p_{i\mathscr{V}} + \sum_{\mathscr{E}\in\mathscr{G}_{i\mathscr{E}}} p_{i\mathscr{E}}, \quad d := \begin{cases} H_i & \text{if } d_i \geq H_i \\ \max(d_i, Ch_i) & \text{otherwise,} \end{cases}$$

where $d_i := \sqrt{\alpha_i/\beta_i}$. Further, let $p_{i\mathscr{W}}$ and $\boldsymbol{p}_{i\mathscr{F}}$ denote the functions f_i and $\boldsymbol{v}_i$, respectively, of Lemma 3. For each subdomain face $\mathscr{F}$ of Ω_i, we then find from Lemmas 1 and 3 and (18) that

$$E_i(\boldsymbol{p}_{i\mathscr{F}}) \le C\omega_i^2 E_i(\boldsymbol{u}_i), \tag{20}$$

where $p_{i\mathscr{F}e} = \nabla p_{i\mathscr{W}e} \; \forall e \in \mathscr{M}_{\mathscr{F}}$ and $p_{i\mathscr{F}e} = 0 \; \forall e \in \mathscr{M}_{\partial\Omega_i} \setminus \mathscr{M}_{\mathscr{F}}$. With reference to (12) and (4), we define

$$\boldsymbol{q}_{i\mathscr{F}} := \sum_{e\in\mathscr{M}_{\mathscr{F}}} q_{ie}\boldsymbol{N}_e, \tag{21}$$

and from elementary finite element estimates and Lemma 4 find

$$\|\boldsymbol{q}_{i\mathscr{F}}\|^2_{L^2(\Omega_i)} \le Ch_i^3 \sum_{e\in\mathscr{M}_{\mathscr{F}}} q_{ie}^2 \le C\|\boldsymbol{q}_i\|^2_{L^2(\Omega_i)} \le C\|\boldsymbol{u}_i\|^2_{L^2(\Omega_i)}, \tag{22}$$

$$\|\nabla\times\boldsymbol{q}_{i\mathscr{F}}\|^2_{L^2(\Omega_i)} \le Ch_i \sum_{e\in\mathscr{M}_{\mathscr{F}}} q_{ie}^2 \le C\|\nabla\times\boldsymbol{u}_i\|^2_{L^2(\Omega_i)}. \tag{23}$$

It follows from Lemmas 2 and 4 that there exists a $\boldsymbol{\Psi}_{i\mathscr{F}} \in (W^{h_i}_{\text{grad}})^3$ such that $\boldsymbol{\Psi}_{i\mathscr{F}} = \boldsymbol{\Psi}_i$ at all nodes of $\mathscr{F}$, that vanishes at all other nodes of $\partial\Omega_i$, and

$$\|\boldsymbol{\Psi}_{i\mathscr{F}}\|^2_{L^2(\Omega_i)} \le C\|\boldsymbol{\Psi}_i\|^2_{L^2(\Omega_i)} \le C\|\boldsymbol{u}_i\|^2_{L^2(\Omega_i)}, \tag{24}$$

$$\|\nabla\times\boldsymbol{\Psi}_{i\mathscr{F}}\|^2_{H^1(\Omega_i)} \le C\omega_i^2\|\boldsymbol{\Psi}_i\|^2_{H^1(\Omega_i)} \le C\omega_i^2\|\nabla\times\boldsymbol{u}_i\|^2_{L^2(\Omega_i)}. \tag{25}$$

From Lemmas 1 and 4, we obtain

$$\|\boldsymbol{\Psi}_i\|^2_{L^2(\partial\mathscr{F})} \le C\omega_i\|\boldsymbol{\Psi}_i\|^2_{H^1(\Omega_i)} \le C\omega_i\|\nabla\times\boldsymbol{u}_i\|^2_{L^2(\Omega_i)}. \tag{26}$$

Let $\boldsymbol{\Psi}_{i\partial\mathscr{F}} \in (W^{h_i}_{\text{grad}})^3$ be identical to $\boldsymbol{\Psi}_i$ at all nodes of $\partial\mathscr{F}$ and vanish at all other nodes of Ω_i. For $\boldsymbol{g} := \Pi^{h_i}(\boldsymbol{\Psi}_{i\partial\mathscr{F}})$, we define

$$\boldsymbol{g}_{i\mathscr{F}} := \sum_{e\in\mathscr{M}_{\mathscr{F}}} g_e^{h_i}\boldsymbol{N}_e. \tag{27}$$

From elementary estimates and (26) we then obtain

$$\|\boldsymbol{g}_{i\mathscr{F}}\|^2_{L^2(\Omega_i)} \le Ch_i^2|\boldsymbol{\Psi}_i\|^2_{L^2(\partial\mathscr{F})} \le C\omega_i h_i^2\|\nabla\times\boldsymbol{u}_i\|^2_{L^2(\Omega_i)}, \tag{28}$$

$$\|\nabla\times\boldsymbol{g}_{i\mathscr{F}}\|^2_{L^2(\Omega_i)} \le C\omega_i\|\nabla\times\boldsymbol{u}_i\|^2_{L^2(\Omega_i)}. \tag{29}$$

Defining

$$\boldsymbol{v}_{i\mathscr{F}} := \nabla p_{i\mathscr{F}} + \boldsymbol{p}_{i\mathscr{F}} + \boldsymbol{q}_{i\mathscr{F}} + \Pi^{h_i}(\boldsymbol{\Psi}_{i\mathscr{F}}) + \boldsymbol{g}_{i\mathscr{F}}, \tag{30}$$

we find $v_{i\mathscr{F}e} = u_{ie} \; \forall e \in \mathscr{M}_{\mathscr{F}}$ and $v_{i\mathscr{F}e} = 0 \; \forall e \in \mathscr{M}_{\partial\Omega_i} \setminus \mathscr{M}_{\mathscr{F}}$. The estimate in (17) then follows from the bounds for each of the terms on the right-hand-side of (30) along with elementary estimates for $\Pi^{h_i}(\boldsymbol{\Psi}_{i\mathscr{F}})$. □

3 BDDC

Background information and related theory for BDDC can be found in several references including [2, 9, 10, 8, 1]. Let u_{Γ_i} and u_Γ denote vectors of finite element coefficients associated with Γ_i and Γ. In general, entries in u_{Γ_i} and u_{Γ_j} are allowed to differ for $j \neq i$ even though they refer to the same finite element edge. Entries in the vector $\tilde{u}_{\Gamma_i}$ are partially continuous in the sense that specific edge values or edge averages over certain subsets of Γ are required to match for adjacent subdomains. In order to obtain consistent entries, we define the weighted average

$$\hat{u}_{\Gamma_i} = R_i \sum_{j=1}^{N} R_j^T D_j \tilde{u}_{\Gamma_j}, \tag{31}$$

where R_j is a 0-1 (Boolean) matrix that selects the rows of u_{Γ_j} from u_Γ and D_j is a diagonal weight matrix with positive entries. The weight matrices form a partition of unity in the sense that

$$\sum_{i=1}^{N} R_i^T D_i R_i = I, \tag{32}$$

where I is the identity matrix. To summarize, $\hat{u}_{\Gamma_i}$ is fully continuous while $\tilde{u}_{\Gamma_i}$ is only partially continuous. The number of continuity constraints that must be satisfied by all the $\tilde{u}_{\Gamma_i}$ determines the dimension of the coarse space.

Let S_i denote the Schur complement associated with Γ_i, which is defined in (39). The system operator for BDDC is the assembled Schur complement

$$S = \sum_{i=1}^{N} R_i^T S_i R_i. \tag{33}$$

From Theorem 25 of [10], the condition number of the BDDC preconditioned operator is bounded above by

$$\kappa(M^{-1}S) \leq \sup_{\tilde{u}_{\Gamma_i}} \frac{\sum_{i=1}^{N} \hat{u}_{\Gamma_i}^T S_i \hat{u}_{\Gamma_i}}{\sum_{i=1}^{N} \tilde{u}_{\Gamma_i}^T S_i \tilde{u}_{\Gamma_i}}. \tag{34}$$

This remarkably simple expression shows that the continuity constraints for $\tilde{u}_{\Gamma_i}$ should be chosen so that large increases in energy do not result from the averaging operation in (31).

For simplicity of notation, we will refer to u_i as the vector of edge finite element coefficients for Ω_i. We have the decomposition

$$u_i = R_{\Gamma_i}^T u_{\Gamma_i} + R_{I_i}^T u_{I_i}, \tag{35}$$

where u_{Γ_i} and u_{I_i} are vectors of coefficients associated with Γ_i and the interior of Ω_i, respectively, and each row of R_{Γ_i} and R_{I_i} has one nonzero entry of unity. We further decompose u_{Γ_i} as

$$u_{i\Gamma} = \sum_{\mathscr{F} \in \mathscr{G}_{i\mathscr{F}}} R_{i\mathscr{F}}^T u_{i\mathscr{F}} + \sum_{\mathscr{E} \in \mathscr{G}_{i\mathscr{E}}} R_{i\mathscr{E}}^T u_{i\mathscr{E}} \tag{36}$$

$$= \sum_{\mathscr{F} \in \mathscr{G}_{i\mathscr{F}}} R_{i\mathscr{F}}^T u_{i\mathscr{F}} + R_{\mathscr{W}_i}^T u_{\mathscr{W}_i} \tag{37}$$

$$= R_{\mathscr{W}_i'}^T u_{\mathscr{W}_i'}^T + R_{\mathscr{W}_i}^T u_{\mathscr{W}_i}, \tag{38}$$

where $\mathscr{W}_i$ denotes the wire basket for Γ_i and $\mathscr{W}_i' = \Gamma_i \setminus \mathscr{W}_i$. The Schur complement associated with Γ_i can be expressed as

$$S_i = A_{\Gamma_i\Gamma_i} - A_{\Gamma_i I_i} A_{I_i I_i}^{-1} A_{I_i \Gamma_i}, \tag{39}$$

where A_i is the stiffness matrix for Ω_i and

$$A_{\Gamma_i\Gamma_i} = R_{\Gamma_i} A_i R_{\Gamma_i}^T, \quad A_{\Gamma_i I_i} = R_{\Gamma_i} A_i R_{I_i}^T, \quad A_{I_i I_i} = R_{I_i} A_i R_{I_i}^T, \quad \text{etc.} \tag{40}$$

Similarly, for $\mathscr{W}_i'$ and $\mathscr{F}$, we introduce the Schur complements

$$S_{\mathscr{W}_i'} = R_{\mathscr{W}_i'} S_i R_{\mathscr{W}_i'}^T, \quad S_{\mathscr{F}_i} = R_{i\mathscr{F}} S_i R_{i\mathscr{F}}^T. \tag{41}$$

Lemma 5 is now rewritten in matrix-vector notation as

$$(R_{i\mathscr{F}} u_{i\Gamma})^T S_{\mathscr{F}_i} (R_{i\mathscr{F}} u_{i\Gamma}) \leq C\omega_i^2 (R_{\mathscr{W}_i'} u_{i\Gamma})^T S_{\mathscr{W}_i'} (R_{\mathscr{W}_i'} u_{i\Gamma}). \tag{42}$$

Because of page restrictions, we only consider a very rich coarse space which includes every edge of each subdomain edge. This coarse space corresponds to Algorithm C of [11]. In this case, we have

$$R_{\mathscr{W}_i} \Delta u_{\Gamma_i} = 0, \tag{43}$$

where $\Delta u := \tilde{u} - \hat{u}$, and it follows from (37) and the positive definiteness of S_i that

$$\Delta u_{\Gamma_i}^T S_i \Delta u_{\Gamma_i} \leq |\mathscr{G}_{i\mathscr{F}}| \sum_{\mathscr{F} \in \mathscr{G}_{i\mathscr{F}}} \Delta u_{i\mathscr{F}}^T S_{\mathscr{F}_i} \Delta u_{i\mathscr{F}}. \tag{44}$$

Let Ω_j denote the subdomain which shares $\mathscr{F}$ with Ω_i, and consider the generalized eigenvalue problem

$$S_{\mathscr{F}_i} \Phi = S_{\mathscr{F}_j} \Phi \Lambda, \tag{45}$$

where Φ is a matrix of eigenvectors normalized so that $\Phi^T S_{\mathscr{F}_j} \Phi = I$ and Λ is a diagonal matrix of positive eigenvalues. Introducing the change of variables $u_{i\mathscr{F}} = \Phi w_{i\mathscr{F}}$, we obtain

$$\Delta u_{i\mathscr{F}}^T S_{\mathscr{F}_i} \Delta u_{i\mathscr{F}} = \Delta w_{i\mathscr{F}}^T \Lambda_m \Delta w_{i\mathscr{F}}, \tag{46}$$

$$\Delta u_{j\mathscr{F}}^T S_{\mathscr{F}_j} \Delta u_{j\mathscr{F}} = \Delta w_{j\mathscr{F}}^T I \Delta w_{j\mathscr{F}}. \tag{47}$$

Choosing

$$\hat{w}_{i\mathscr{F}} = \hat{w}_{j\mathscr{F}} = (\Lambda + I)^{-1} (\Lambda \tilde{w}_{i\mathscr{F}} + \tilde{w}_{j\mathscr{F}}), \tag{48}$$

we find

$$\Delta w_{i\mathscr{F}} = (\Lambda + I)^{-1} (\tilde{w}_{i\mathscr{F}} - \tilde{w}_{i\mathscr{F}}), \tag{49}$$

$$\Delta w_{j\mathscr{F}} = (\Lambda + I)^{-1} \Lambda (\tilde{w}_{j\mathscr{F}} - \tilde{w}_{i\mathscr{F}}), \tag{50}$$

and from (46) and (47) obtain

$$\Delta u_{i\mathscr{F}}^T S_{\mathscr{F}_i} \Delta u_{i\mathscr{F}} + \Delta u_{j\mathscr{F}}^T S_{\mathscr{F}_j} \Delta u_{j\mathscr{F}} \leq 4 (\tilde{u}_{i\mathscr{F}}^T S_{\mathscr{F}_i} \tilde{u}_{i\mathscr{F}} + \tilde{u}_{j\mathscr{F}}^T S_{\mathscr{F}_j} \tilde{u}_{j\mathscr{F}}). \tag{51}$$

From (44), (51) and (42), we obtain

$$\sum_{i=1}^{N} \Delta u_{\Gamma_i}^T S_i \Delta u_{\Gamma_i} \leq C\omega^2 \sum_{i=1}^{N} \tilde{u}_{\Gamma_i}^T S_i \tilde{u}_{\Gamma_i}, \tag{52}$$

where

$$\omega = \max_i 1 + \log(H_i/h_i). \tag{53}$$

Finally, from (34), (52), and the triangle inequality, we obtain

Theorem 1 (Condition Number Estimate). *The condition number of the BDDC preconditioned operator for this study is bounded by*

$$\kappa \leq C\omega^2. \tag{54}$$

In summary, we have obtained a favorable condition number estimate that requires no assumptions on the material properties of the subdomains. We are unaware of any other algorithms for 3D H(curl) problems with this property. Comparing the condition number estimate of Theorem 1 with those in (1) and (2), we see that the factor of $H_i^2\beta_i/\alpha_i$ has been removed and the logarithmic factor has been reduced from four powers to two. We note that the estimate in Theorem 1 also holds for FETI-DP due its spectral equivalence with BDDC.

The algorithm involves a change of variables for edges of each subdomain face, and the choice for $\hat{w}_{i\mathscr{F}}$ and $\hat{w}_{j\mathscr{F}}$ in (48) corresponds to the diagonal weight matrices

$$R_{i\mathscr{F}} D_i R_{i\mathscr{F}}^T = \Lambda(\Lambda + I)^{-1}, \tag{55}$$

$$R_{i\mathscr{F}} D_j R_{i\mathscr{F}}^T = (\Lambda + I)^{-1}. \tag{56}$$

We note this change of variables can be implemented in practice with just a few simple modifications to the standard BDDC algorithm. Referring back to the discussion before (46), the change of variables can be expressed as

$$u_{i\Gamma} = T_i w_{i\Gamma}.$$

Notice that rows of the square transformation matrix T_i for edges not on a subdomain face will have a single diagonal entry of unity since no change of variables is made for those edges, while the rows of T_i corresponding to subdomain face $\mathscr{F}$ are obtained from the matrix of eigenvectors Φ appearing in (45). One can then replace D_j in (31) by $\tilde{D}_j := T_j D_{jc} T_j^{-1}$, where D_{jc} is the diagonal weight matrix associated with the new variables (see (56)). In terms of the algorithm in [2], the changes amount to replacing W_i in (16) and (19) by $\tilde{D}_i$ and W_i in (18) and (20) by $\tilde{D}_i^T$. The importance of the change of variables for some problems is shown in the next section.

4 Numerical Results

In this section, we present some numerical results to verify the theory and also to provide some additional insights. The domain is a unit cube discretized into smaller cubic elements. All the examples are solved to a relative residual tolerance of 10^{-8} for random right-hand-sides using the conjugate gradient algorithm with BDDC as the preconditioner. The number of iterations and condition number estimates from conjugate gradients are under the headings of *iter* and *cond* in the tables. We consider three different types of weights for the averaging operator. The first one, designated *eig*, is the one of the previous section based on a change of variables and the solution of an eigenproblem. Unless otherwise specified in the tables, this is the weighting used. The second type, *stiff*, is based on a conventional approach in which the weights are proportional to entries on the diagonals of subdomain matrices. The third, *card*, uses the inverse of the cardinality of an edge, i.e. the reciprocal of the number of subdomains sharing the edge, for the weight.

The results in Table 1 are consistent with theory, suggesting condition numbers are bounded independently of the number of subdomains, while the results in Table 2 are consistent with the $\log(H/h)^2$ estimate of Theorem 1.

We also consider a checkerboard distribution of material properties in which (α, β) for a subdomain is either (α_1, β_1) or (α_2, β_2), and note that subdomains with the same properties are connected together only at their corners. Results for 64 cube subdomains each with $H/h = 4$ are shown in Table 3. Notice for only one choice of material properties in the table that all three types of weighting lead to small condition numbers, and only the *eig* approach always gives condition numbers which are independent of the material properties. We also investigated another type of weighting similar to *card*, but with weights γ, $0<\gamma<1$ for faces of subdomains with properties α_1,β_1 and $1-\gamma$ for faces of subdomains with properties α_2,β_2. Regardless of the choice of γ, large condition numbers were observed for the properties in the final row of Table 3. We note also that the choice of material properties in the final row is not covered by the theory of [11].

In the final example, we consider a cube mesh of 20^3 elements that is partitioned into different numbers of subdomains using the graph partitioner Metis [7]. Although this example is not covered by our theory because the subdomains have irregular shapes, the results in Table 4 indicate that the algorithm of this study continues to perform well. The results in Tables 3 and 4 suggest that the *eig* weighting of this study may be necessary in order to effectively solve problems with material property jumps or with subdomains having irregular shapes.

Table 1. Results for N cube subdomains, each with $\beta=1$ and $H/h=4$.

N	$\alpha=10^2$ iter (cond)	$\alpha=1$ iter (cond)	$\alpha=10^{-2}$ iter (cond)
4^3	15 (2.70)	14 (2.63)	10 (1.77)
6^3	16 (2.88)	15 (2.81)	11 (2.05)
8^3	16 (2.95)	15 (2.87)	12 (2.23)
10^3	17 (2.98)	16 (2.91)	13 (2.33)

Table 2. Results for 64 cube subdomains, each with $\beta=1$.

H/h	$\alpha=10^2$ iter (cond)	$\alpha=1$ iter (cond)	$\alpha=10^{-2}$ iter (cond)
4	15 (2.70)	14 (2.63)	10 (1.77)
6	17 (3.30)	16 (3.21)	11 (2.14)
8	18 (3.77)	16 (3.66)	13 (2.46)
10	19 (4.16)	18 (4.03)	13 (2.72)

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Table 3. Checkerboard material property results for 64 cube subdomains with $H/h = 4$.

α_1	β_1	α_2	β_2	*eig* iter (cond)	*stiff* iter (cond)	*card* iter (cond)
1	1	10^3	1	10 (1.59)	19 (4.57)	196 (1.64e3)
1	1	1	10^3	11 (1.96)	84 (2.69e2)	109 (4.72e2)
1	1	1	1.01	14 (2.63)	14 (2.63)	14 (2.63)
10^2	10^{-2}	1	1	6 (1.07)	65 (3.17e2)	74 (1.65e2)

Table 4. Results for 20^3 elements partitioned into N subdomains using a graph partitioner. Material properties are constant with $\alpha = 1$ and $\beta = 1$.

N	*eig* iter (cond)	*stiff* iter (cond)	*card* iter (cond)
60	19 (4.30)	189 (6.31e2)	24 (9.06)
65	19 (4.40)	184 (6.34e2)	29 (1.55e3)
70	18 (3.89)	188 (6.47e2)	23 (7.48)
75	19 (4.16)	176 (6.12e2)	23 (6.49)

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