Basic Algorithms, Assignment 4, Solutions
Due, Thursday, Oct 4

MIDTERM October 9 In Class!
Wednesday, October 3: NO OFFICE HOURS
SPECIAL PREMIDTERM OFFICE HOURS Monday, October 9, 7-8:30p.m.

1. When \( H \) is a heap with length fifty million and \texttt{HEAPIFY-UP(A,300)} is called what is the maximum number of exchanges that can take place. What is the minimiml number of exchanges that can take place.
Solution. Minimal is zero, when \( A(300) \) is already bigger than its parent. Then maximum is if you exchange up to the root with 150, 75, 37, 18, 9, 4, 2, 1 for 8 exchanges.

2. When \( H \) is a heap with length fifty million and \texttt{HEAPIFY-DOWN(A,300)} is called what is the maximum number of exchanges that can take place. What is the minimil number of exchanges that can take place.
Solution. Minimal is zero, when \( A(300) \) is already bigger than its children. Then maximum is if you exchange down to the leaf with 600, 1200, 2400, \ldots\. You would do this \( \lfloor \log_{2} \left( \frac{50000000}{300} \right) \rfloor = 17 \) times.

3. Consider a heap \( H \) with length 1023.\footnote{Did you recognize 1023 as a special number? Its one less than 1024 = 2^{10}. The binary tree with that many nodes just fills out a row!} Assume the elements of the array are distinct. Let \( x \) be the third largest element in the array. What are the possible positions for \( x \).
Solution: It can be anywhere on the second or third levels but nowhere else so it can be in positions 2, 3, 4, 5, 6, 7.
Let \( y = H[700] \). Can \( y \) be the largest element in the array?
Solution: Yes, the largest element can be anywhere in the last row.
Can \( y \) be the smallest element in the array?
Solution: No, thats always at the top in position one.
What is the smallest \( i \) so that it is possible that \( y \) is the \( i \)-th smallest element of the array.
Its ancestors must be smaller than it but thats it. Node \( x \) has \( \lfloor \log_{2} x \rfloor \) ancestors (or you can just enumerate them for 700: 350, 175, \ldots\ so that \( i = 9 \).
4. Page 69, Problem 8. (Do part (b) only for $k = 3$.)
Solution: (With two jars) Drop the first jar from $x, 2x, 3x, \ldots$ until in
crashes. When it crashes between $ix$ and $(i+1)x$ do the second jar
one by one between them. So the first jar is used at most $N/x$ times
and the second one at most $x$ times for a total of $N/x + x$ times. Some
calculus gives that it is best to take $x = \sqrt{N}$ and then the total drops
is $2\sqrt{N}$ which is certainly $o(N)$.

With three jars again drop the first jar from $x, 2x, 3x, \ldots$ until in
crashes. This takes $N/x$ drops. Now we have $x$ possibilities and we
are in the two jar case so we can do the rest with $2\sqrt{x}$ drops. The
total is $N/x = 2\sqrt{x}$. Some calculus gives that it is best to take $x = N^{1/3}
and then the total drops is $3N^{1/3}$ which is $o(\sqrt{N})$ so the third jar
helped. This pattern continues.

A person who never made a mistake never tried anything new.
– Albert Einstein