The Power of Friction: Quantifying the "Goodness" of Frictional Grasps *

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Abstract

This paper attempts to quantify the trade-offs that exist among the *number of fingers*, *coefficient of friction* and the *the goodness of a grasp*. In particular, we give a general framework for defining a grasp metric that takes friction into account. Our approach rectifies a flaw in an older metric that was proposed by Ferrari and Canny [FC92] for the frictional case. In addition, we discuss several algorithmic solutions related to frictional grasps.

1 Introduction

A standard approach to grasp theory takes the point of view that it suffices to consider the situation where the fingers make only *frictionless point contacts* with the object to be grasped. Under such a model, referred to as a *positive grip* model, one then considers various grasp existence and synthesis problems. (See Mishra, Schwartz and Sharir [MSS87] for several key results under this model.) This point of view is justified by the argument that presence of friction only improves the grasp and hence positive grips represent in some sense the most pathological situation. Another argument favoring the positive grip model is that even when there is uncertainty about how much friction is available, the grasps synthesized under this restricted model remain immune to such uncertainties. However, there have been relatively inadequate attempts to answer many questions that the above viewpoint raises.

For instance, given a frictionless closure grasp, precisely how much does the presence of friction improves that grasp? In order to answer such a question, one needs to define a quantitative measure for a goodness of a grasp and then provide a numerical measure of how a grasp improves in terms of the coefficient of Coulomb friction μ as it varies from 0 (frictionless situation) to ∞ (unbounded amount of friction). In the last few years, starting with the work of Kirkpatrick et. al. [KMY92] dealing with the positive closure grasps, there have been several "grasp metrics" proposed by many researchers[1] (see, for instance, [FC92, Mis94, Tei95b, Tei95a]). However, only the work of Ferrari and Canny directly addresses the problem of defining grasp metrics for the frictional contact models. Unfortunately, their metrics for this case turns out to be flawed. For instance, in some situations, their metric leads one to conclude that for sufficiently large coefficient of friction a grasp may be less desirable than the one that involves no friction!

In this paper, we provide a further generalization of Mishra's approach [Mis94] to include friction and show that this new class of metrics measures goodness of a grasp faithfully. In particular, it does not suffer from the sorts of problems that Ferrari-Canny metric does.

Another related interesting problem raised by the frictional model is that of the synthesis of closure grasps with relatively small number of fingers. Markenskoff et. al. [MNP90] showed that in the presence of friction it is always possible to synthesize 3 and 4 finger closure grasps for planar and 3D objects, respectively. These numbers are also lower bounds.

We also consider a generalization of a problem first addressed by Schwartz and Sharir.

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[1] The paper by Mishra [Mis94] provides a unified framework for defining all of these metrics. The recent work of Teichmann [Tei95b, Tei95a] rectifies a problem with most of these metrics and proposes a class of metrics that do not depend on the reference frame.

Given k "feasible gripping" points on an object and a positive real number $r > 0$, determine the least amount of friction $\mu^*(r)$ necessary in order that some subset of the gripping points will yield a good closure grasp of quality r or larger.

We provide a polynomial-time algorithm to solve this problem approximately.

2 The Formal Framework

We start with a highly idealized robot hand model. It consists of several independently movable force-sensing stiff fingers; this hand is used to grasp a rigid object B. The fingers are placed at points $\mathbf{p}$ of the boundary of B, which we shall denote by ∂B. Additionally, we make the following assumptions: (1) B is a full-bodied (i.e. no internal holes) compact subset of the Euclidean 3-space, and has a piece-wise smooth boundary ∂B. (2) For each finger-contact on the body, we may associate a nominal point of contact, $\mathbf{p} \in \partial B$. We let ∂B^* denote the set of points $\mathbf{p} \in \partial B$ such that the direction $\mathbf{n}(\mathbf{p})$ normal to ∂B at $\mathbf{p}$ is well-defined; by convention, we pick $\mathbf{n}(\mathbf{p})$ to be the unit normal pointing into the interior of B.

For each such point $\mathbf{p}$, we can define a wrench system $\{\mathbf{\Gamma}^{(1)}(\mathbf{p}), \mathbf{\Gamma}^{(2)}(\mathbf{p}), \ldots, \mathbf{\Gamma}^{(m)}(\mathbf{p})\}$, $(0 \le m \le 6)$, where the number and screw-axes of the wrench system depend on the contact type. Some of these wrenches can be *bisense* (i.e. can act in either sense) and the remaining wrenches, *unisense*.

In the case, where the contacts are frictionless, we call the corresponding grips 'positive grips.' The wrench system associated with each point is:

$$\mathbf{\Gamma}(\mathbf{p}) = \{[\mathbf{n}(\mathbf{p}), \mathbf{p} \times \mathbf{n}(\mathbf{p})]\}$$

In general, corresponding to a set of finger-contacts, we have a system of n wrenches,

$$\{\mathbf{w}_1, \ldots, \mathbf{w}_k, \mathbf{w}_{k+1}, \ldots, \mathbf{w}_n\},$$

the first k of which are bisense and the remaining last $n-k$ of the wrenches are unisense. Assume that the magnitudes of these wrenches are given by the scalars f_i's

$$\{f_1, \ldots, f_k, f_{k+1}, \ldots, f_n\},$$

where $f_1, \ldots, f_k \in \mathbb{R}$ and $f_{k+1}, \ldots, f_n \in \mathbb{R}_{\ge}$, and not all the magnitudes are zero. We call such a system of wrenches and the wrench-magnitudes, a *grip*, G, and say that this grip G generates an external wrench $\mathbf{w} = [F_x, F_y, F_z, \tau_x, \tau_y, \tau_z] \in \mathbb{R}^6$, if

$$\mathbf{w} = \sum_{i=1}^{n} f_i \ \mathbf{w}_i.$$

This leads to the concept of a *closure grasp*:

Definition 2.1 *A set of gripping points on an object B to which corresponds a system of wrenches $\mathbf{w}_1, \ldots, \mathbf{w}_n$ (as before) is said to constitute a* force/torque closure grasp *(or simply, a* closure grasp*) if and only if any arbitrary external wrench can be generated by varying the magnitudes of the wrenches (subject to the constraints imposed by the senses of the wrenches).*

A necessary and sufficient condition for a closure grasp is that the (module) sum of the linear space spanned by the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_k$ and the positive space spanned by the vectors $\mathbf{w}_{k+1}, \ldots, \mathbf{w}_n$ is the entire $\mathbb{R}^6$:

$$\text{lin }(\mathbf{w}_1, \ldots, \mathbf{w}_k) + \text{pos }(\mathbf{w}_{k+1}, \ldots, \mathbf{w}_n) = \mathbb{R}^6.$$

Denote, by L, the linear space lin $(\mathbf{w}_1, \ldots, \mathbf{w}_k)$, and, by $L^\perp$, the orthogonal complement of L in $\mathbb{R}^6$. Let π be the linear projection function of $\mathbb{R}^6$ onto $L^\perp$ whose kernel is L. A necessary and sufficient condition for a closure grasp is equivalently given as follows:

$$\mathbf{0} \in \text{int conv }(\pi\mathbf{w}_{k+1}, \ldots, \pi\mathbf{w}_n)$$

in $L^\perp$. In case of positive grips ($k = 0$), we have, $\mathbf{0} \in$ int conv $(\mathbf{w}_1, \ldots, \mathbf{w}_n)$.

A simple linear time algorithm for finding *at least* one set of force targets that can generate a given external wrench has been presented in [MSS87]. Also, as the external wrench is varied in the course of a manipulation task, this algorithm updates the force targets in constant time. A more extensive discussion on the structure of the wrench map and how this structure can be exploited in grasp synthesis appears in Teichmann's thesis [Tei95a].

2.1 Introducing Friction

The number of fingers necessary for a closure grasp can be reduced in the presence of friction. Usually the Coulomb friction model for the surface contacts is assumed (see, for example, [KR86]). A friction cone is associated with each contact, and the line of action of the force transmitted through the contact must lie within this cone.

Let $\mu > 0$ be the friction coefficient. Closure with n fingers under friction occurs when any arbitrary external wrench $\mathbf{w}$ can be expressed as

$$\mathbf{w} = \sum_{i=1}^{n} f_i \Big[\mathbf{\Gamma}(\mathbf{p}_i) + t_i(\mathbf{n}_i^{\perp}, \mathbf{p}_i \times \mathbf{n}_i^{\perp}) \Big], \qquad (1)$$

where $f_i \geq 0$ and $\mathbf{n}_i^{\perp}$ is a unit normal to $\mathbf{n}(\mathbf{p}_i)$ with $0 \leq t_i \leq \mu$, $(1 \leq i \leq n)$. In other words, the actual force applied at point $\mathbf{p}_i$ is a non-negative multiple of $\mathbf{n}(\mathbf{p}_i) + t_i \mathbf{n}_i^{\perp}$ and belongs to the friction cone $\mathcal{C}_\mu(\mathbf{p}_i)$, a set of vectors forming an angle of at most $\arctan(\mu)$ with $\mathbf{n}(\mathbf{p}_i)$. Define the set of forces of unit magnitude in the friction cone

$$\mathcal{F}_\mu(\mathbf{p}) = \{\mathbf{f} \in \mathcal{C}_\mu(\mathbf{p}) : ||\mathbf{f}|| = 1\}.$$

We now show that the standard techniques for testing the closure properties still apply. We would like to say that since the finger can apply a force anywhere in the friction cone, we can include the image by pos $\mathbf{\Gamma}$ of the entire friction cone in the set of forces that can be generated at the contact point. Let $\mathbf{\Gamma}(\mathbf{p}, \mathbf{f}) = [\mathbf{f}, \mathbf{p} \times \mathbf{f}]$ and define

$$\mathbf{\Gamma}_\mu(\mathbf{p}) = \mathbf{\Gamma}(\mathbf{p}, \mathcal{F}_\mu(\mathbf{p})). \qquad (2)$$

Note that $\mathbf{\Gamma}_0(\mathbf{p}) = \{\mathbf{\Gamma}(\mathbf{p})\}$. We require the force applied from any direction to be of unit magnitude (in contrast with [FC92]) so that $f\mathbf{\Gamma}_\mu(\mathbf{p})$ represents the effect of applying a force of magnitude f at $\mathbf{p}$. This avoids certain problems when evaluating grasp strength as discussed earlier.

To generate an external wrench $\mathbf{w}$, we would like to select points from $\bigcup_i$ pos $\mathbf{\Gamma}_\mu(\mathbf{p}_i)$ each point corresponding to a finger position, direction and magnitude of applied force. There are two potential difficulties. First, to a point in wrench space may correspond two or more fingers. The second difficulty is that we are allowed to select *at most one* point (actually ray) for each pos $\mathbf{\Gamma}_\mu(\mathbf{p}_i)$ since a finger can apply a force only in one direction at any one time. We show that this is always possible.

For $S \subseteq \mathbb{R}^d$, denote by $[0,1]S$ the set $\{\lambda s \in S : 0 \leq \lambda \leq 1\}$.

Lemma 2.1 *Both* pos $\mathbf{\Gamma}_\mu(\mathbf{p})$ *and* $[0,1]\mathbf{\Gamma}_\mu(\mathbf{p}, \mathbf{n})$ *are convex.*

Proof. To show the first statement, let $\mathbf{w}_i = [\mathbf{f}_i, \mathbf{p}_i \times \mathbf{f}_i] \in$ pos $\mathbf{\Gamma}_\mu(\mathbf{p})$, $i = 1, 2$. For $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$,

$$\lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 = [\lambda_1 \mathbf{f}_1 + \lambda_2 \mathbf{f}_2, \mathbf{p} \times (\lambda_1 \mathbf{f}_1 + \lambda_2 \mathbf{f}_2)].$$

But $\lambda_1 \mathbf{f}_1 + \lambda_2 \mathbf{f}_2 \in \mathcal{F}_\mu(\mathbf{p})$ by convexity of the friction cone, which proves the first statement. Since $\lambda \mathbf{\Gamma}_\mu(\mathbf{p}, \mathbf{n}) = \mathbf{\Gamma}_\mu(\mathbf{p}, \lambda \mathbf{n})$, if in addition $||\mathbf{f}_i|| \leq 1$, then we have $||\lambda_1 \mathbf{f}_1 + \lambda_2 \mathbf{f}_2|| \leq \lambda_1 + \lambda_2 = 1$ by the triangle inequality and the second statement follows. □

In two dimensions, the friction cone $\mathcal{C}_\mu(\mathbf{p})$ is bounded by two rays emanating from the origin and its image in wrench space $\mathbf{\Gamma}_\mu(\mathbf{p})$ is a planar cone ($\mathbf{\Gamma}$ being linear) and also bounded by two rays. For three dimensional objects however, the friction cone is a circular cone in three dimension with apex at the origin. Its image under $\mathbf{\Gamma}$ is a convex cone with apex at the origin, but is no longer linear. An early analysis of the image of a friction cone was done by Ji [Ji87].

We state two generalized versions of Carathéodory's theorem (without proof), which we call *thickened* versions[2].

Theorem 2.2 *Consider a family of convex sets A_1, A_2, $\ldots, A_n$ in $\mathbb{R}^d$ and a point $x \in$ conv $(A_1, \ldots, A_n)$. Then there exist $m = \min\{n, d+1\}$ points $y_1, y_2, \ldots, y_m$, each belonging to a unique A_i such that $x \in$* conv $\{y_1, \ldots, y_m\}$. □

There is a constructive procedure to compute the y_i's, given an appropriate description of the convex sets A_i; the algorithms takes total time of $O(nd)$, when the A_i's are polytopes given by their vertices. We also have the following

Theorem 2.3 *Consider a family of convex sets A_1, A_2, $\ldots, A_n$ in $\mathbb{R}^d$ and a point $x \in$ conv $(0, A_1, \ldots, A_n)$. Then there exist $m = \min\{n, d\}$ points $y_1, y_2, \ldots, y_m$, each belonging to a unique A_i such that $x \in$* conv $\{0, y_1, \ldots, y_m\}$. □

Applying theorem 2.3 to the sets pos $\mathbf{\Gamma}_\mu(\mathbf{p}_i)$, which are convex by lemma 2.1, we conclude that only one point from each is necessary to generate an external wrench (which is called x in the theorem.) Hence $\bigcup_i$ pos $\mathbf{\Gamma}_\mu(\mathbf{p}_i)$ does indeed represent the set of possible wrenches the fingers can generate. Therefore, to test for closure in the frictional setting, we can check if the origin is in the interior of the convex hull of the friction cones associated with the finger contacts. This is however computationally non-trivial, as pos $\mathbf{\Gamma}_\mu(\mathbf{p}_i)$ has non-linear boundaries, at least for three dimensional objects. In

[2] A different generalization called a 'multiplied version' is due to Bárány.

practice, an approximation is usually used; this is examined more closely in the full paper. (See [Tei95a].) The test for closure can then be implemented as in the non-frictional case using such an approximation.

3 Efficiency of a Closure Grasp

In the description of closure grasps, there is an implicit unrealistic assumption that *the magnitudes of finger forces are no way constrained.* In particular, it is quite likely that a closure grasp may resist any arbitrary external wrench; but it may only do so by applying an unrealistically large force at a finger in response to a fairly small external wrench in some direction.

In order to alleviate this problem, we may assume that certain additional constraint is imposed on the magnitudes of the finger forces—the "*finger force constraint*" being expressible as

$$\begin{aligned}\chi &: \mathbb{R}^n \to \{0,1\}\\ &: (f_1,\ldots,f_n) \mapsto \begin{cases} 1, & \text{if the "constraint" holds;}\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Let $G = \{\mathbf{p}_1,\ldots,\mathbf{p}_n\}$ be a grasp. The set of external wrenches that can be generated by the grasp, subject to the finger force constraint, χ, is given by $\mathcal{W}_\chi$, called the "*feasible wrench set*:"

$$\begin{aligned}&\mathcal{W}_\chi(\boldsymbol{\Gamma}_\mu(\mathbf{p}_1),\ldots,\boldsymbol{\Gamma}_\mu(\mathbf{p}_n))\\ &= \left\{\mathbf{w} = \sum_{i=1}^{n} f_i \boldsymbol{\Gamma}_\mu(\mathbf{p}_i) : \chi(f_1,\ldots,f_n) = 1\right\} \subseteq \mathbb{R}^6.\end{aligned}$$

We also call $-\mathcal{W}_\chi$ the "*resistable wrench set*," the set of external wrenches that can be resisted by the grasp.

Some natural finger force constraints that one may impose are of the following kinds:

<u>Convex Constraint</u>: This constraint allows us to bound on the total force applied by all the fingers.

$$\chi_{con} : f_1 \geq 0,\ldots,f_n \geq 0 \text{ and } \sum_{i=1}^{n} f_i \leq 1.$$

Thus, $\mathcal{W}_{\chi_{con}} = \mathrm{conv}\{[0,1]\boldsymbol{\Gamma}_\mu(\mathbf{p}_i) : 1 \leq i \leq n\}$.

<u>Max Constraint</u>: It corresponds to the case where each finger can apply a bounded force, independently of the others.

$$\chi_{max} : f_1 \geq 0,\ldots,f_n \geq 0 \text{ and } \max_{i\in\{1,\ldots,n\}} f_i \leq 1.$$

$\mathcal{W}_{\chi_{max}}$ is given by the Minkowski sum of the vectors $\boldsymbol{\Gamma}_\mu(\mathbf{p}_1)$, ..., $\boldsymbol{\Gamma}_\mu(\mathbf{p}_n)$:

$$\mathcal{W}_{\chi_{max}} = \bigoplus_{i=1}^{n} \{[0,1]\boldsymbol{\Gamma}_\mu(\mathbf{p}_i) : 1 \leq i \leq n\}.$$

<u>Hybrid Constraint</u>: This constraint corresponds for example to the case where we have several robot arms and wish to bound the total forces on each arm separately. Let P_1, P_2, ..., P_l be a partition of the indices $\{1, \ldots, n\}$. Then

$$\chi_{hyb} : f_1 \geq 0,\ldots,f_n \geq 0 \text{ and } \sum_{i\in P_j} f_i \leq 1,\ 1 \leq j \leq l.$$

Thus

$$\begin{aligned}\mathcal{W}_{\chi_{hyb}} &= \bigoplus_{j=1}^{l} \mathrm{conv}\{[0,1]\boldsymbol{\Gamma}_\mu(\mathbf{p}_i) : i \in P_j\}\\ &= \mathrm{conv}\bigoplus_{j=1}^{l}\{[0,1]\boldsymbol{\Gamma}_\mu(\mathbf{p}_i) : i \in P_j\}\end{aligned}$$

For $\mu > 0$, we need to show that with the new constraints on the finger forces, we can still express an external wrench $\mathbf{v} \in -\mathcal{W}_\chi(\boldsymbol{\Gamma}_\mu(\mathbf{p}_1),\ldots,\boldsymbol{\Gamma}_\mu(\mathbf{p}_n))$ as a sum $\mathbf{v} = \sum_{i=1}^{n} f_i \mathbf{w}_i$ with $\chi(f_1,\ldots,f_n) = 1$ and $\mathbf{w}_i \in \boldsymbol{\Gamma}_\mu(\mathbf{p}_i)$.

For the case of χ_{con}, this follows from theorem 2.3 applied to $[0,1]\boldsymbol{\Gamma}_\mu(\mathbf{p}_i)$, $i = 1,\ldots,n$, which are convex sets by the second statement of lemma 2.1. For the case of χ_{max}, this follows from the definition of Minkowski sum. A combination of the reasonings used in these two cases, can be used to prove a similar result for χ_{hyb} also.

3.0.1 Grasp Quality Measures

For any set $X \subseteq \mathbb{R}^d$, let the *residual ball* of X refer to the maximal ball $\mathcal{B}(X)$ centered at the origin 0 such that $\mathcal{B}(X)$ is fully contained inside the convex hull of X. The *residual radius* of X, denoted $r(X)$ is the radius of this residual ball $\mathcal{B}(X)$. Note that

$$\mathcal{W}_{\chi_{con}} \subseteq \mathcal{W}_{\chi_{hyb}} \subseteq \mathcal{W}_{\chi_{max}} \subseteq n\, \mathcal{W}_{\chi_{con}}$$

and $r_{\chi_{con}} \leq r_{\chi_{hyb}} \leq r_{\chi_{max}} \leq n\ r_{\chi_{con}}$. The corresponding residual radii are our grasp metrics under the appropriate constraints[3].

[3] We note that, for example in the frictionless case, the complexity of the Minkowski sum of the segments $\overline{\mathbf{0}\boldsymbol{\Gamma}(\mathbf{p}_i)}$ is much larger than the convex hull of those segments since every segment contributes more than one vertex on the Minkowski sum. This Minkowski sum is known as a *zonotope*. In the six dimensional case, the upper bound on its complexity is $O(n^5)$, while for the convex hull, it is $O(n^3)$.

3.1 The Case of Friction

Allowing friction at the finger contacts causes the set of applicable forces to increase. Thus it is not surprising that grasp strength, as measured by the measures described here also increases. In fact some grasps which are not closure without friction, become closure grasps for a sufficiently large coefficient of friction. This is the case for example of two fingers placed at antipodal points on a planar object. Figure 1 shows a four

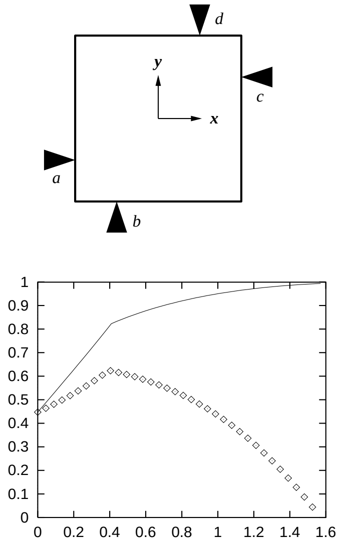

Figure 1: A four finger grasp and its strength as a function of the friction angle. The continuous curve corresponds to the grasp metric defined here, where as the dotted curve corresponds to the one due to Ferrari and Canny. The dotted curve approaches of zero for large friction even when the grasp remains a closure grasp.

finger closure grasp in the plane, call it G_4, and a graph showing grasp strength as measured by r_{con} as a function of the *friction angle*, the maximum angle a force can deviate from the object normal at the contact point, while maintaining contact. Corresponding to this grasp G_4, we also show in figure 2 the 'top' view (from a point on the positive torque axis) of the set conv $\mathbf{\Gamma}(G_4)^4$.

3.1.1 An Alternative Formulation

A different method of approximating the image of a friction cone under the wrench map has been used in the

[4] The figures for this section were generated using our residual radius implementation. Our software uses the QHULL library and the GEOMVIEW program of the Geometry Center, University of Minnesota.

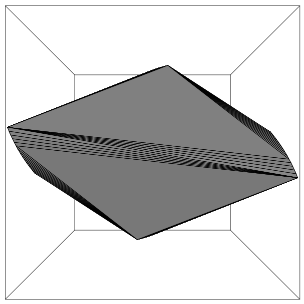

Figure 2: The grasp of figure 1 for friction coefficient 0.2.

literature for the purpose of computing grasp strength. Consider planar objects. It is possible to simply take the convex hull of image under $\mathbf{\Gamma}$ of the edges bounding the friction cone, as done in [FC92]. However this may cause a significant underestimate of the quality of a grasp.

Let us consider for example the following somewhat contrived situation. A square is being grasped with four fingers placed at a, b, c, d, as in figure 1. Fingers at b and d are frictionless, while finger at $a = (-2, -1)$ and $c = (2, 1)$ have a coefficient of friction μ. Let $\phi = \arctan \mu$ be the corresponding half-angle of the friction cone. The point a maps for example to two points a^+ and a^- in the wrench space. We have

$$a^{\pm} = [\cos\phi, \pm\sin\phi, \mp 2\sin\phi - \cos\phi].$$

Then as μ is increased, the projections onto the xy-plane of a^+ and a^- move close to the projection of $\mathbf{\Gamma}(d)$ and $\mathbf{\Gamma}(b)$ and the radius of the largest ball centered at the origin inside conv $\{a^{\pm}, b^{\pm}, \mathbf{\Gamma}(c), \mathbf{\Gamma}(d)\}$ decreases for sufficiently large angle ϕ. This is illustrated by the dotted curved in figure 1 (bottom). Thus for sufficiently large friction coefficient, increasing friction actually decreases grasp quality according to this simplified measure. Other examples and difficulties are discussed in Teichmann's thesis [Tei95a]. Note, however, that the approximation is reasonably accurate for small μ, at least in the planar case.

4 Computational Issues

4.1 Approximating Friction cones

The main difficulty in computing these measures, not to mention performing grasp optimization based on them, is that conv $\mathbf{\Gamma}_\mu(\mathbf{p})$ has non-linear boundary for $\mu > 0$. It is customary in the robotics literature to approximate the boundary of the friction cone by a polyhedral cone.

This is sufficient for detecting the presence or absence of closure. For the purpose of computing grasp strength, this is not sufficient, and causes the grasp strength to be underestimated.

We have already defined $\mathbf{\Gamma}_\mu$ in equation (2) to be the image by $\mathbf{\Gamma}$ of all *unit* forces a finger can apply at a given point. This set lies on $\mathcal{S}^2 \oplus \mathbb{R}^3$ or on $\mathcal{S}^1 \oplus \mathbb{R}$ for planar objects. We can approximate this set as follows. We start by approximating the friction cone $\mathcal{C}_\mu$ by N_1 regularly spaced rays. Similarly, we approximate the part of the truncated friction cone $\mathcal{F}_\mu$ that lies on $\mathcal{S}^2$. We obtain a N_2-vertex polyhedron S_{N_2} approximating a unit sphere. This polyhedron is then intersected with the approximation of the unbounded friction cone; the total complexity of the intersection is $O(N_1 + N_2)$. Let us denote by $\mathbf{\Gamma}(N_1, N_2)$ the resulting approximation. After taking all such approximations for each finger, one can compute an approximate value $\tilde{r}$ of the grasp strength simply by solving at most M linear programming problems as in Kirkpatrick et. al. [KMY92]. A careful analysis shows that:

$$\frac{r_{(\mu-\varepsilon)}}{1+\delta} \leq \tilde{r} \leq r_\mu.$$

In the equation, ε depends on N_1 and δ, on N_2. Using this technique, we can solve the following problem:

> Given k "feasible gripping" points on an object and a positive real number $r > 0$, one can compute an approximate value for the coefficient of friction $\tilde{\mu}$ which guarantees that some subset of the gripping points will yield a good closure grasp of quality r or larger. Furthermore, if $\mu^*(r)$ represents the necessary optimal value, then
>
> $$\tilde{\mu} - \mu^*(r[1+\delta]) \leq \varepsilon.$$

Of course, $\mu^*(r) \leq \tilde{\mu}$. The value $\tilde{\mu}$ satisfying the approximation requirements can be computed in time linear in k and polynomial in $1/\varepsilon$ and $1/\delta$.

4.1.1 Generating Force Targets

Consider an external wrench $\mathbf{v}$ being applied to B which is being grasped by m fingers with frictional contacts. Mishra et. al. [MSS87] give a linear time procedure that given a set W of vectors $\mathbf{w}_i = \mathbf{\Gamma}(\mathbf{p}_i)$ in $\mathbb{R}^6$, $i = 1, \ldots, m$, finds a subset of 6 of them such that $\mathbf{v}$ can be expressed as a positive combination of those vectors. This can be used for finding force targets for frictionless fingers. The coefficients returned by their procedure however do not necessarily satisfy the finger force constraints.

We now describe a linear time procedure for finding force targets in the case of χ_{con} constraint. The problem discussed earlier can be handled as a result of the following

Theorem 4.1 *Given a set of points $p_1, \ldots, p_m \in \mathbb{R}^d$, with $0 \in \operatorname{conv}\{p_1, \ldots, p_m\}$, and a point $v \in \operatorname{conv}\{p_1, \ldots, p_m\}$. Then one can find $\lambda_1, \ldots, \lambda_m$, with $0 \leq \lambda_i \leq 1$, and $\sum_{i=1}^m \lambda_i = 1$ at most d of which are non-zero, such that $v = \sum_{i=1}^m \lambda_i p_i$.* □

The theorem has a constructive counterpart, leading to an algorithm based on linear programming and some simple techniques to compute barycentric coordinates. The details appear in [Tei95a]. We use this algorithm for the case where $d = 6$ and the p_i's correspond to the wrenches $\mathbf{w}_i$'s and v to the external wrench $\mathbf{v}$. In the frictional case, one can use the above algorithm applied to an approximation of the friction cones as in section § 3, then "merge" any two points selected from the friction cones as per theorem 2.3. This might be sufficient in practice.

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(Additional references appear in the full paper.)