Grasp Metrics: Optimality and Complexity

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In this paper, we discuss and compare various metrics for goodness of a grasp. We study the relations and trade-offs among the goodness of a grasp, geometry of the grasped object, number of fingers and the computational complexity of the grasp-synthesis algorithms. The results here employ the techniques from convexity theory first introduced by the author and his colleagues [14, 21].

1 Introduction

In robotics, the hand models usually consist of a small number (four or five) of articulated fingers, possibly with a palmar surface. A variety of finger arrangements have been suggested: for example, an anthropomorphic arrangement consisting of an opposable 'thumb' and several more fingers arranged to work cooperatively.

Many problems in *dextrous manipulation* involving such a hand model has been studied by mapping an object to be manipulated into a low dimensional hypersurface in a higher-dimensional wrench space by the so-called "*wrench map* [21]." By studying the convexity geometric properties of the image of this wrench map, one can answer several interesting questions in this field.

- Lower and upper bounds on the number of fingers to grasp an object under a variety of models [18, 21]. The arguments employ various Helly-type theorems: namely, Carathéodory's theorem [5] and Steinitz's theorem [30].
- Properties of *force* and *form closures* and a related geometric notion of *immobility*. Relation between force and form closures [18, 22, 23].
- Characterization of ungraspable ("exceptional") objects [21].
- Linear time (thus, optimal) grasp synthesis algorithms [21].
- Grasp control: updating the finger forces in $O(1)$ time in response to a time-varying external wrench [18].
- Study of the strength (also called, efficiency) of a grasp [9, 14]. The algorithms and the bounds are based on a quantitative version of the Steinitz's theorem. Also, see [2]
- Analysis and planning algorithms for fixturing and workholding [3, 19, 20, 35].

Several other approaches to these problems have emerged, contemporaneously or subsequently. Notable among these approaches are purely geometric techniques [4, 10, 16, 25, 31], topological techniques [12] and algebraic techniques [28, 29]. However, while in certain instances these other techniques have proven to be more powerful, they seem to lack the depth and elegance of the convexity theoretic approach.

In this paper, we take a fresh look at the problem of formulating "*grasp metrics*" and their relation with the quantitative versions of Helly-type theorems. Parts of the results described here are based on some old results (jointly with Kirkpatrick and Yap) and some work in progress (jointly with Teichmann).

2 Terminology and Overview

Formally, we consider an idealized robot hand, consisting of several independently movable force-sensing fingers; this hand is used to grasp a rigid object B. Furthermore, we make the following simplifying assumptions:

- (*Smooth Body*) B is a full-bodied (i.e. no internal holes) compact subset of the Euclidean 3-space. Furthermore, B has a piece-wise smooth boundary ∂B.
- (*Point Contact*) For each finger-contact on the body, we may associate a nominal point of contact, $\mathbf{p} \in \partial B$. By convention, we pick $\mathbf{n}(\mathbf{p})$ to be the unit normal pointing into the interior of B.

  For each such point $\mathbf{p}$, we can define a wrench system $\{\Gamma^{(1)}(\mathbf{p}), \Gamma^{(2)}(\mathbf{p}), \ldots, \Gamma^{(\ell)}(\mathbf{p})\}$, $(0 \leq \ell \leq 6)$, where the number and screw-axes of the wrench system depend on the contact type. Some of these wrenches can be *bisense* (i.e. can act in either sense) and the remaining wrenches, *unisense*. (For a discussion of screw theory, and in particular, wrenches and twists, see [13] and [26]. Also, see the appendix.)
- (*Compliance*) We will consider the case when the fingers are stiff—the force/torques applied at the fingers are generated by some actuators whose mechanics need not concern us.

Many interesting special cases occur, depending on how we model the *static friction* and the *stiction* between the fingers and the body B. In the case, where the contacts are frictionless, a finger can only apply force $\mathbf{f}$ on the body in the direction $\mathbf{n}(\mathbf{p})$ at the point $\mathbf{p}$. Also if the fingers are non-sticky, then the force $\mathbf{f}$ has a non-negative magnitude, $f = \mathbf{f} \cdot \mathbf{n}(\mathbf{p}) \geq 0$. Such grips are also known as 'positive grips'. In this case, the wrench system associated with each point is:

$$\Gamma(\mathbf{p}) = \{[\mathbf{n}(\mathbf{p}), \mathbf{p} \times \mathbf{n}(\mathbf{p})]\}$$

Thus, corresponding to a set of finger-contacts, we have a system of n wrenches,

$$\{\mathbf{w}_1, \ldots, \mathbf{w}_k, \mathbf{w}_{k+1}, \ldots, \mathbf{w}_n\},$$

the first k of which are bisense and the remaining last $n-k$ of the wrenches are unisense. Let us assume that the magnitudes of these wrenches are given by the scalars f_i's

$$\{f_1, \ldots, f_k, f_{k+1}, \ldots, f_n\},$$

where $f_1, \ldots, f_k \in \mathbb{R}$ and $f_{k+1}, \ldots, f_n \in \mathbb{R}_{\geq 0}$, and not all the magnitudes are zero. We call such a system of wrenches and the wrench-magnitudes, a *grip*, $\mathcal{G}$, and say that this grip $\mathcal{G}$ generates an external wrench $\mathbf{w} = [F_x, F_y, F_z, \tau_x, \tau_y, \tau_z] \in \mathbb{R}^6$, if

$$\mathbf{w} = \sum_{i=1}^{n} f_i\, \mathbf{w}_i.$$

In matrix notation, the above equation is expressed as

$$\mathbf{w} = \mathcal{W} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix},$$

where $\mathcal{W}$ is a $6 \times n$ matrix whose columns are the corresponding n wrenches of the system $\{\mathbf{w}_1, \ldots, \mathbf{w}_k, \mathbf{w}_{k+1}, \ldots, \mathbf{w}_n\}$, associated with the contact points of the grip. The matrix $\mathcal{W}$ is called a *grip matrix* of the grip defining the system of wrenches

$$\{\mathbf{w}_1, \ldots, \mathbf{w}_k, \mathbf{w}_{k+1}, \ldots, \mathbf{w}_n\}.$$

2.1 Closure Grasp

Next, we consider the concept of a *closure grasp*: A system of wrenches $\mathbf{w}_1, \ldots, \mathbf{w}_n$ (as before) is said to constitute a *force/torque closure grasp* if and only if any arbitrary external wrench can be generated by varying the magnitudes of the wrenches (subject to the constraints imposed by the senses of the wrenches). A necessary and sufficient condition for a closure grasp is that the (module) sum of the linear space spanned by the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_k$ and the positive space spanned by the vectors $\mathbf{w}_{k+1}, \ldots, \mathbf{w}_n$ is the entire $\mathbb{R}^6$:

$$\text{lin}\ (\mathbf{w}_1, \ldots, \mathbf{w}_k) + \text{pos}\ (\mathbf{w}_{k+1}, \ldots, \mathbf{w}_n) = \mathbb{R}^6.$$

Let us denote, by L, the linear space lin $(\mathbf{w}_1, \ldots, \mathbf{w}_k)$, and, by $L^{\perp}$, the orthogonal complement of L in $\mathbb{R}^6$. Let π be the linear projection function of $\mathbb{R}^6$ onto $L^{\perp}$ whose kernel is L. Then it can be shown that a necessary and sufficient condition for a closure grasp is

$$\text{lin }(\mathbf{w}_1, \ldots, \mathbf{w}_k) + \text{pos }(\pi\mathbf{w}_{k+1}, \ldots, \pi\mathbf{w}_n) = \mathbb{R}^6.$$

The above equation in turn is equivalent to the following conditions:

$$\mathbf{0} \in \text{int conv }(\pi\mathbf{w}_{k+1}, \ldots, \pi\mathbf{w}_n)$$

in $L^\perp$. Here, if $k = 0$ (i.e. positive grip) then the above condition reduces to the following:

$$\mathbf{0} \in \text{int conv }(\mathbf{w}_1, \ldots, \mathbf{w}_n).$$

Let us assume that $\dim(L) = d$. Then there is a linear basis W of L

$$W = \{\mathbf{w}_{j_1}, \ldots, \mathbf{w}_{j_d}\} \subseteq \{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$$

which when adjoined with a set of vectors W',

$$W' = \left\{\mathbf{w}_{j_{d+1}}, \ldots, \mathbf{w}_{j_6}\right\} \subseteq \{\mathbf{w}_{k+1}, \ldots, \mathbf{w}_n\}$$

yields $\widehat{W} = W \cup W'$, a linear basis of $\mathbb{R}^6$. Thus under the condition that we have a closure grasp, we can find $\mathbf{g} \in \mathbb{R}^k \times \mathbb{R}^{n-k}_{\geq 0}$ such that

$$-(\mathbf{w}_{k+1} + \cdots + \mathbf{w}_n) = \sum_{i=1}^{n} g_i\ \mathbf{w}_i,$$

and thus

$$\sum_{i=1}^{n} f_{h,i}\ \mathbf{w}_i, = \mathbf{0},$$
$$\text{where } \mathbf{f}_h \in \mathbb{R}^k \times \mathbb{R}^{n-k}_{\geq 0},$$
$$\text{and } f_{h,k+1} > 0, \ldots, f_{h,n} > 0.$$

In other words,

$$\mathbf{f}_h \in \text{null}(\mathcal{W}) \cap \mathbb{R}^k \times \mathbb{R}^{n-k}_{>0};$$

i.e., $\mathbf{f}_h$ is a null vector of the grip matrix and all its unisense components are strictly positive.

Now any external wrench $\mathbf{w}$ can be expressed as a linear combination of the vectors in the basis $\widehat{W}$. Thus there is a vector $\mathbf{f}_p \in \mathbb{R}^n$, whose non-zero entries are in the positions $j_1, \ldots, j_6$, and

$$\mathbf{w} = \sum_{k=1}^{6} f_{p,j_k}\ \mathbf{w}_{j_k} = \sum_{i=1}^{n} f_{p,i}\ \mathbf{w}_i.$$

Now consider a vector $\mathbf{f} = \mathbf{f}_p + \lambda\ \mathbf{f}_h \in \mathbb{R}^k \times \mathbb{R}^{n-k}_{\geq 0}$, where λ is chosen to be of a sufficiently large positive value that ensures that the negative components in $\mathbf{f}_p$ are dominated by the positive components of $\mathbf{f}_h$. Thus,

$$\mathbf{w} = \sum_{i=1}^{n} f_i\ \mathbf{w}_i = \sum_{i=1}^{n} (f_{p,i} + \lambda \cdot f_{h,i})\ \mathbf{w}_i.$$

These arguments yield a simple algorithm to find *at least* one set of force targets that can generate a given external wrench. Also, as the external wrench is varied in the course of a manipulation task, a slight variation of this algorithm updates the force targets in $O(1)$ time.

Observe that the above formulation has turned a problem in mechanics into a combinatorial geometric problem, now amenable to many interesting techniques in convexity theory and computational and combinatorial geometry.

2.2 Grasp Metric: Desiderata

Given a grasp $\mathcal{G}$ described by a wrench system

$$\{\mathbf{w}_1, \ldots, \mathbf{w}_n\},$$

we would frequently like to be able to say how good this grasp is as compared to another grasp producing a different wrench system. Clearly, such a "measure of goodness" must possess some physical intuitions that correspond to how we normally view a grasp—e.g., a closure grasp should be preferable to a non-closure grasp (i.e., immobile grasp) or a force/torque closure grasp should be better than one achieving only force closure but not torque closure, etc. However, the situation becomes more complicated as we begin to compare two grasps that are both force-torque closure—here, we must rely on the underlying physics. Independent of the physical considerations, however, it seems appropriate that the grasp metrics possess some simple mathematical properties enumerated below.

In what follows, we shall refer to the value of a grasp under a grasp metric by the terms *quality*, *strength* and/or *efficiency*, in view of the nonstandard and often confusing arrays of terminologies that have appeared in the literature, and denote such a metric by r with a distinguishing subscript. By abuse of notation, sometimes we will call a "measure" a grasp metric, even if it does not satisfy the following properties. In this paper, a grasp metric will yield a positive scalar real value and the grasps will be totally ordered, although, we foresee situations where a grasp metric may be given by a higher dimensional vector (e.g., a pair) with some partial ordering on the grasps. The generalizations are straightforward. The grasp metric can also be generalized to include certain other parameters (e.g, friction, stiction, tasks under considerations, scaling in various force/torque dimensions, hand geometry, some universality criteria, etc.). We simply avoid these added complications at this point.

Let the grasp, $\mathcal{G}$, be given by the wrench system

$$\{\mathbf{w}_1, \ldots, \mathbf{w}_n\},$$

and let the associated grasp strength be given by

$$r = r(\mathcal{G}) = r(\mathbf{w}_1, \ldots, \mathbf{w}_n).$$

Then the following properties hold.

1. **Positivity:** For any grasp $\mathcal{G}$ with the wrench system
$$\{\mathbf{w}_1, \ldots, \mathbf{w}_n\},$$
$r = r(\mathcal{G}) \geq 0$, the inequality being strict if and only if $\mathcal{G}$ is a force/torque closure grasp.

2. **Boundedness:** If the cardinality of the wrench system, $n < \infty$ is bounded (i.e., the number of contact points for $\mathcal{G}$ is bounded), then so is $r = r(\mathcal{G}) < \infty$. This condition is weaker than the corresponding condition that requires r to be bounded, independent of n.

3. **Subadditivity:** For any two grasps $\mathcal{G}_1$ and $\mathcal{G}_2$, with the wrench systems:
$$\{\mathbf{w}_1, \ldots, \mathbf{w}_m\},$$
and
$$\{\mathbf{w}_{m+1}, \ldots, \mathbf{w}_n\},$$
respectively, we have
$$\begin{aligned} & r(\mathbf{w}_1, \ldots, \mathbf{w}_m, \mathbf{w}_{m+1}, \ldots, \mathbf{w}_n) \\ & \quad \geq \; r(\lambda \mathbf{w}_1, \ldots, \lambda \mathbf{w}_m) \\ & \qquad\qquad + r((1-\lambda)\mathbf{w}_{m+1}, \ldots, (1-\lambda)\mathbf{w}_n), \end{aligned}$$
where $0 \leq \lambda \leq 1$.

An immediate consequence of the above axioms is that, for a given object and a given hand with fixed number of fingers (with associated contact types), if the object allows a force/torque closure grasp then there is an optimal grasp of the object with that hand with a grasp quality r^*, when the grasp metric satisfies the first two properties.

Additional consequence of the above axioms are as follows:

1. **Scaling:** For any $\lambda > 1$,
$$r(\lambda \mathbf{w}_1, \ldots, \lambda \mathbf{w}_n) \geq r(\mathbf{w}_1, \ldots, \mathbf{w}_n).$$
However, many of our grasp metrics can be shown to actually satisfy the equality condition under scaling.

2. **Monotonicity:**
$$r(\mathbf{w}_1, \ldots, \mathbf{w}_n, \mathbf{w}_{n+1}) \geq r(\mathbf{w}_1, \ldots, \mathbf{w}_n).$$
Also, note that
$$\begin{aligned} & r(\mathbf{w}_1, \ldots, \mathbf{w}_m, \mathbf{w}_{m+1}, \ldots, \mathbf{w}_n) \\ & \quad \geq \; \max(r(\mathbf{w}_1, \ldots, \mathbf{w}_m), r(\mathbf{w}_{m+1}, \ldots, \mathbf{w}_n)). \end{aligned}$$
Thus the last condition tells us that a hand with large number of fingers is better than another with smaller number of fingers, assuming that they allow same contact types.

2.3 Grasp Metrics Based on Resistable Wrench

Note that in the above descriptions of closure grasps, we have made an implicit unrealistic assumption that *the magnitudes of finger forces are no way constrained.* In particular, it is quite likely that a force/torque closure grasp may resist any arbitrary external wrench; but it may only do so by applying an unrealistically large force at a finger in response to a fairly small external wrench in some direction.

In order to alleviate this problem, we may assume that certain additional constraint is imposed on the magnitudes of the finger forces—the "*finger force constraint*" being expressible as

$$\begin{aligned} \chi &: \mathbb{R}^n \to \{0,1\} \\ &: (f_1, f_2, \ldots, f_n) \mapsto \begin{cases} 1, & \text{if the "constraint" holds;} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

The characteristic function naturally defines the set

$$\begin{aligned} S_\chi &= \Big\{(f_1, f_2, \ldots, f_n) \in \mathbb{R}^n : \\ &\qquad \chi(f_1, f_2, \ldots, f_n) = 1\Big\} \\ &\subseteq \mathbb{R}^n. \end{aligned}$$

We say that χ (or equivalently, S_χ) is *faithful* if

$$\mathbb{R}^k \times \mathbb{R}^{n-k}_{\geq 0} \subseteq \text{pos}\ (S_\chi),$$

and that χ is *strongly faithful* if

$$\mathbb{R}^k \times \mathbb{R}^{n-k}_{\geq 0} = \text{pos}\ (S_\chi),$$

Thus the set of external wrenches that can be generated by the grasp, subject to the finger force constraint, χ, is given by G_χ, called the "*feasible wrench set:*"

$$\begin{aligned} &G_\chi(\mathbf{w}_1, \ldots, \mathbf{w}_n) \\ &= \Big\{\mathbf{w} = \sum_{i=1}^n f_i \mathbf{w}_i : \chi(f_1, f_2, \ldots, f_n) = 1\Big\} \\ &\subseteq \mathbb{R}^6. \end{aligned}$$

We also use the notation $R_\chi = -G_\chi$ to denote the "*resistable wrench set,*" the set of external wrenches that can be resisted by the grasp.

Note that

- If S_χ is convex, closed and compact, then so is G_χ.
- If $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ forms a force/torque closure grasp and if χ is faithful then
$$\mathbf{0} \in \text{int}\ G_\chi(\mathbf{w}_1, \ldots, \mathbf{w}_n).$$
- If $S_{\chi_1} \subset S_{\chi_2}$ then $G_{\chi_1} \subset G_{\chi_2}$.

Some natural finger force constraints that one may impose are of the following kinds:

- **Convex Constraint**:
$$\begin{aligned} \chi_{con} &: f_{k+1} \geq 0, \ldots, f_n \geq 0 \\ &\quad \text{and} \sum_{i=1}^n |f_i| \leq 1. \end{aligned}$$
  Note that χ_{con} is convex, closed, compact and strongly faithful.

  $G_{\chi_{con}}$ is given by the convex hull of the vectors $\mathbf{w}_1$, $-\mathbf{w}_1$, ..., $\mathbf{w}_k$, $-\mathbf{w}_k$, $\mathbf{w}_{k+1}$, ..., $\mathbf{w}_n$:
$$\begin{aligned} G_{\chi_{con}} &= \text{conv}\Big(\{\mathbf{w}_i, -\mathbf{w}_i : 1 \leq i \leq k\} \\ &\qquad \cup \{\mathbf{w}_i : k+1 \leq i \leq n\}\Big). \end{aligned}$$

- **Max Constraint**:
$$\begin{aligned} \chi_{max} &: f_{k+1} \geq 0, \ldots, f_n \geq 0 \\ &\quad \text{and} \max_{i \in \{1, \ldots, n\}} |f_i| \leq 1. \end{aligned}$$
  Clearly, χ_{max} is convex, closed, compact and strongly faithful.

  $G_{\chi_{max}}$ is given by the Minkowski sum of the vectors $\mathbf{w}_1$, $-\mathbf{w}_1$, ..., $\mathbf{w}_k$, $-\mathbf{w}_k$, $\mathbf{w}_{k+1}$, ..., $\mathbf{w}_n$:
$$\begin{aligned} G_{\chi_{max}} &= \bigoplus\Big(\{\mathbf{w}_i, -\mathbf{w}_i : 1 \leq i \leq k\} \\ &\qquad \cup \{\mathbf{w}_i : k+1 \leq i \leq n\}\Big). \end{aligned}$$

- **Hybrid Constraint**:
  Let P_1, P_2, ..., P_l be a partition of the indices $\{1, \ldots, n\}$. Then

$$\chi_{hyb} \quad : \quad f_{k+1} \geq 0, \ldots, f_n \geq 0 \quad \text{and} \sum_{i \in P_j} |f_i| \leq 1, \quad 1 \leq j \leq l.$$

  Again, χ_{hyb} is convex, closed, compact and strongly faithful.

  $G_{\chi_{hyb}}$ is given by the Minkowski sum of the convex hulls of the vectors corresponding to each partition P_j:

$$G_{\chi_{hyb}} = \bigoplus_{j=1}^{l} \text{conv}\Big(\{\mathbf{w}_i, -\mathbf{w}_i : i \in P'_j\} \cup \{\mathbf{w}_i : i \in P''_j\}\Big),$$

  where

$$P'_j = P_j \cap \{1, \ldots, k\},$$

  and

$$P''_j = P_j \cap \{k+1, \ldots, n\}.$$

Now consider the largest ball of radius $r = r_\chi(\mathbf{w}_1, \ldots, \mathbf{w}_n)$ in $\mathbb{R}^6$ centered at $\mathbf{0}$ and contained in the corresponding feasible wrench set, $G_\chi(\mathbf{w}_1, \ldots, \mathbf{w}_n)$. We shall refer to this r as the "*residual radius*" of G_χ. Then it is trivial to see that there exists an external wrench of magnitude only infinitesimally larger than r that cannot be generated or resisted by the grasp under consideration, if it must respect the finger force constraint χ. This value r may thus be used to define a grasp metric.

Note that since

$$S_{\chi_{con}} \subseteq S_{\chi_{hyb}} \subseteq S_{\chi_{max}} \subseteq n\, S_{\chi_{con}},$$

we have

$$G_{\chi_{con}} \subseteq G_{\chi_{hyb}} \subseteq G_{\chi_{max}} \subseteq n\, G_{\chi_{con}}$$

and

$$r_{\chi_{con}} \leq r_{\chi_{hyb}} \leq r_{\chi_{max}} \leq n\, r_{\chi_{con}}.$$

Note that, since the underlying geometric problem remains largely unchanged irrespective of the finger force constraint chosen, we shall often focus only on the simplest situation represented by the constraint χ_{con}.

2.4 Grasp Metrics: Variations

(1) One may consider a finger force constraint of the following kind:

$$\tilde{\chi}_2 : \sum_{i=1}^{n} (f_i)^2 \leq 1.$$

Note that $\tilde{\chi}_2$ is convex, closed, compact and faithful, but not strongly faithful, as $S_{\tilde{\chi}_2} = \mathcal{B}^n$, the n-dimensional ball, and

$$\text{pos}\,(S_{\tilde{\chi}_2}) = \mathbb{R}^n.$$

Then $G_{\tilde{\chi}_2}$ is the image of the n-dimensional ball, $\mathcal{B}^n$, under the linear map defined by the grip matrix $\mathcal{W}$, and thus a 6-dimensional ellipsoid. Since the lengths of the principal axes of the resulting ellipsoid are given by the singular values of the grip matrix $\mathcal{W}$ (i.e., the nonnegative square roots of the eigenvalues of the real positive definite square matrix $\mathcal{W}^T\mathcal{W}$), its residual radius is given by the smallest singular value of the grip matrix $\mathcal{W}$, $r_{\tilde{\chi}_2} = \sigma_{min}(\mathcal{W}) > 0$. Note that, since $\tilde{\chi}_2$ is faithful but not strongly faithful, this grasp metric may be highly misleading. However, note that

$$r_{\chi_2} < r_{\tilde{\chi}_2},$$

where χ_2 is a corresponding strongly faithful constraint of the following form:

$$\chi_2 \quad : \quad f_{k+1} \geq 0, \ldots, f_n \geq 0 \text{ and } \text{and} \sum_{i=1}^{n} (f_i)^2 \leq 1.$$

Note, however that r_{χ_2} can be arbitrarily small in relation to $r_{\tilde{\chi}_2}$.

(2) Another grasp metric was suggested by Jeff Trinkle [33]. This metric will be denoted here by r_{null} and the motivations for it are described below. Note that earlier we had observed that, given a grasp with the corresponding system of n wrenches,

$$\{\mathbf{w}_1, \ldots, \mathbf{w}_k, \mathbf{w}_{k+1}, \ldots, \mathbf{w}_n\},$$

that satisfy the closure condition, we always have a nontrivial null vector $\mathbf{f}_h$ of the grip matrix $\mathcal{W}$, $\mathbf{f}_h = \{f_{h,1},$

$\ldots, f_{h,k}, f_{h,k+1}, \ldots, f_{h,n}\} \in \mathbb{R}^k \times \mathbb{R}^{n-k}_{>0}$ (all unisense components are strictly positive) such that

$$\mathbf{0} = \sum_{i=1}^{n} f_{h,i}\ \mathbf{w}_i = \mathcal{W} \begin{bmatrix} f_{h,1} \\ f_{h,2} \\ \vdots \\ f_{h,n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Following Trinkle's suggestion, we now consider the grasp metric defined as follows:

$$r_{null} = \max_{\mathbf{f} \in \text{null}(\mathcal{W})} \Big\{ \min_{k+1 \le i \le n} f_i : \sum_{i=1}^{n} f_i\ \mathbf{w}_i = \mathbf{0} \ \ \&\ \ \mathbf{f} \in \mathbb{R}^k \times \mathbb{R}^{n-k}_{>0} \Big\}.$$

The quality r_{null} can be computed efficiently by the following linear programming formulation:

- GIVEN: A $6 \times n$ grip matrix $\mathcal{W}$ and a linear, compact, closed, convex and strongly faithful finger force constraint condition χ.
- SOLVE:

$$\begin{array}{lrcl} \text{maximize} & \lambda & & \\ \text{subject to} & \mathcal{W}\mathbf{f} & = & \mathbf{0} \\ & -\lambda & \le & 0 \\ & \lambda - f_i & \le & 0, \\ & & & (k+1 \le i \le n) \\ & \chi(\mathbf{f}) & = & 1, \end{array}$$

where $\mathbf{f} = (f_1, \ldots, f_n)^T$ and λ is a real number.

For instance, if the finger force constraint χ is χ_{con}, then the last condition is simply

$$f_{k+1} \ge 0, \ldots, f_n \ge 0,$$

and

$$|f_1| + \cdots + |f_k| + |f_{k+1}| + \cdots + |f_n| \le 1.$$

For a closure grasp there is a positive λ satisfying the feasibility conditions, and thus, since the feasible set is bounded, there is an optimal solution: $r_{null} = \lambda^*$, $\mathbf{f}_h^* = (f_{h,1}^*, \ldots, f_{h,k}^*, f_{h,k+1}^*, \ldots, f_{h,n}^*)$.

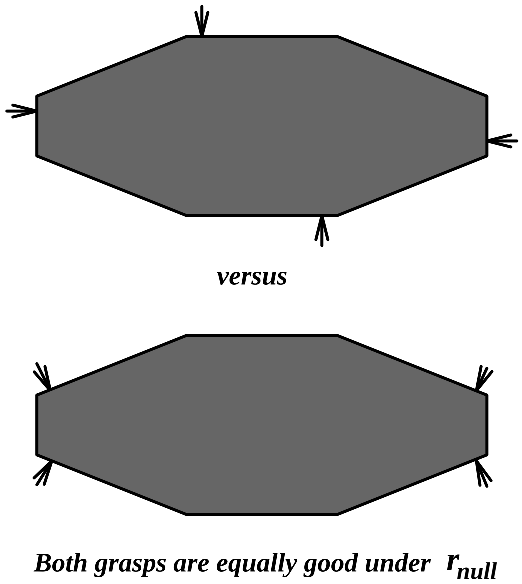

Figure 1: *The grasp metric r_{null} does not distinguish between the above two grasps.*

Note that r_{null} satisfies the positivity and boundedness conditions but not the monotonicity (thus, subadditivity) property. That is adding a new finger can actually make an existing grasp worse! For instance, in the Figure 1, only considering the force-closure, it is easily seen that r_{null} for both the grasps are same (i.e., the largest possible value 1/4).

Here, we shall compute certain relations that explicitly exhibit certain problems with the grasp metric r_{null}. Recall that, we have a subset $\widehat{W} \subset \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$, linearly spanning the wrench space $\mathbb{R}^6$. Among all such $\widehat{W}$'s, let $\widehat{W}^*$ be a basis of $\mathbb{R}^6$ that is *maximally orthonormal* in the sense that it maximizes the following positive real-valued function:

$$d(\widehat{W}) = \left(\frac{|\det \widehat{W}|\ (\min_{\mathbf{w} \in \widehat{W}} |\mathbf{w}|)}{\prod_{\mathbf{w} \in \widehat{W}} |\mathbf{w}|} \right),$$

where $\det \widehat{W}$ is the determinant of a 6×6 square matrix whose columns are the vectors of $\widehat{W}$. Note that by

Hadamard inequality, we have

$$0 \leq d(\widehat{W}^*) \leq \min_{\mathbf{w} \in \widehat{W}} |\mathbf{w}|.$$

We show that

$$r_{\chi_{con}} \geq d(\widehat{W}^*) \left(\frac{r_{null}}{1 + 6 r_{null}} \right),$$

where r_{null} is computed with the linear condition χ_{con}.

We proceed as follows: Consider an external wrench $\mathbf{w}$ oriented arbitrarily but of magnitude

$$|\mathbf{w}| = d(\widehat{W}^*) \left(\frac{r_{null}}{1 + 6 r_{null}} \right).$$

It suffices to show that $\mathbf{w}$ can be generated by the wrenches in the system of wrenches associated with the grasp, subject to the finger force constraint. Let $\widetilde{\mathbf{w}} = (1 + 6 r_{null})\mathbf{w}$. Clearly, $\widetilde{\mathbf{w}}$ can be expressed as a linear combination of the vectors in $\widehat{W}^*$ as follows:

$$\widetilde{\mathbf{w}} = \sum_{k=1}^{6} f_{p,j_k} \mathbf{w}_{j_k},$$

such that by Cramer's rule,

$$f_{p,j_k} = \frac{\det \left[\mathbf{w}_{j_1}, \ldots, \mathbf{w}_{j_{k-1}}, \widetilde{\mathbf{w}}, \mathbf{w}_{j_{k+1}}, \ldots, \mathbf{w}_{j_6} \right]}{\det \widehat{W}^*},$$

and

$$\begin{aligned} |f_{p,j_k}| &\leq \frac{|\mathbf{w}_{j_1}| \cdots |\mathbf{w}_{j_{k-1}}| \, |\widetilde{\mathbf{w}}| \, |\mathbf{w}_{j_{k+1}}| \cdots |\mathbf{w}_{j_6}|}{|\det \widehat{W}^*|} \\ &= r_{null} \frac{\min_{\mathbf{w} \in \widehat{W}^*} |\mathbf{w}|}{|\mathbf{w}_{j_k}|} \\ &\leq r_{null}. \end{aligned}$$

Thus, we can express $\widetilde{\mathbf{w}}$ as

$$\begin{aligned} \widetilde{\mathbf{w}} &= \sum_{k=1}^{6} f_{p,j_k} \mathbf{w}_{j_k} + \sum_{i=1}^{n} f^*_{h,i} \mathbf{w_i} \\ &= \sum_{i=1}^{n} f_i \mathbf{w_i}, \end{aligned}$$

where $f_{k+1} \geq 0$, ..., $f_n \geq 0$, since $|f^*_{h,i}| \geq r_{null}$ and $|f_{p,j_k}| \leq r_{null}$. However,

$$\sum |f_i| \leq 1 + 6 r_{null},$$

since $\sum |f^*_{h,i}| \leq 1$ and $\sum |f_{p,j_k}| \leq 6 r_{null}$. Thus by scaling, we have

$$\mathbf{w} = \frac{\widetilde{\mathbf{w}}}{1 + 6 r_{null}} = \sum_{i=1}^{n} \frac{f_i}{1 + 6 r_{null}} \mathbf{w_i},$$

satisfying all the necessary conditions for χ_{con}. Thus

$$r_{\chi_{con}} \geq |\mathbf{w}| = d(\widehat{W}^*) \left(\frac{r_{null}}{1 + 6 r_{null}} \right).$$

However, in general, a large value for r_{null} does not imply a good value for $r_{\chi_{con}}$, if we do not have a good value for $d(\widehat{W}^*)$. For instance, for any dimension d (e.g., $d = 6$), there is a positive $\epsilon_d > 0$, such that for any $0 < \epsilon < \epsilon_d$, we can always find $d + 1$ unit vectors, $\{\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_d\}$, such that the corresponding $r_{\chi_{con}} = \epsilon$, but $r_{null} > 1/(2d)$. Choose $\mathbf{w}_0$ arbitrarily, and place the remaining d vectors closely in a cluster such that their centroid is on the ray $\lambda(-\mathbf{w}_0)$ (where $\lambda > 0$) and the d simplex contains a residual d-ball of radius ϵ—this can always be accomplished. Then there is a small $0 < \delta < 1 - 1/d$ such that

$$\left(\frac{1 - \delta}{2} \right) \mathbf{w}_0 + \left(\frac{1 + \delta}{2d} \right) (\mathbf{w}_1 + \cdots + \mathbf{w}_d) = \mathbf{0}.$$

Conversely, for any dimension d, and any sufficiently small $\epsilon > 0$, we can find $n = \lceil 1/\epsilon \rceil$ d-dimensional unit vectors $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$, such that $r_{null} \leq \epsilon$ but $r_{\chi_{con}} \geq 1/d$. Assume that $n = m(d + 1)$ is a multiple of $d + 1$. Choose $d + 1$ clusters of m unit vectors each and place each cluster closely about the vertices of a regular d-simplex Δ inscribed within the unit sphere. Since the convex hull of these vectors contains Δ, it has a residual radius of value no smaller than $1/d$, but since $r_{null} \leq 1/n$, r_{null} is arbitrarily small.

(3) Yet another variation on this theme may be obtained by considering some different "geometric object" inscribed in the feasible wrench set, G_χ. Easy variations can be obtained by either considering full-dimensional spheres with respect to different norms (i.e. L_1 or L_∞ norms, instead of L_2), ellipsoids (the parameters being dependent on the task that the hand is supposed to perform) or by considering lower dimensional spheres or ellipsoids (i.e., 3-spheres spanning

only the force vectors, and/or 3-spheres spanning only the torque vectors.)

(4) Another special case arises as follows: Suppose we know that the grasp is required to resist a set of external wrenches, each of which can be expressed as a sum of a fixed external wrench $\overline{\mathbf{w}}$ and an additional arbitrarily varying external wrench $\tilde{\mathbf{w}}$, whose magnitude and orientation are unknown. We wish to maximize the magnitude of this unknown component to the extent possible; the associated grasp metric is then given by this maximal value. It is easily seen that the grasp metric is given by the radius of the largest sphere centered at $-\overline{\mathbf{w}}$ and contained in the feasible wrench set G_χ. An ubiquitous example of such a constant external wrench is given by the wrench generated by the weight of the object being grasped. Sometimes, we may further generalize this concept by requiring that the constant external wrenches are known only to the extent that they belong to a set $\overline{\mathbf{W}}$. Then we wish to maximize the parameter r such that $\overline{\mathbf{W}} \oplus r\mathcal{B} \subseteq G_\chi$, where $\overline{\mathbf{W}} \oplus r\mathcal{B}$ is the Minkowski sum of the set $\overline{\mathbf{W}}$ and a 6-dimensional ball $\mathcal{B}$ of radius r.

(5) One important special case is as follows: Recall that the map Γ applied to a point $\mathbf{p} \in \partial B$ produces a system of wrenches in $\mathbb{R}^6$ that would result if a unit force is applied at the contact point $\mathbf{p}$. Also recall that the map Γ is uniquely defined by the contact type and the point on the boundary. Let us now consider the set $\Gamma(\partial B) \subseteq S^2 \oplus \mathbb{R}^3$ (the unit cylinder). Let $\lambda \in \mathbb{R}_{\geq 0}$ be a maximal positive real number such that

$$\lambda\Gamma(\partial B) \subseteq G_\chi.$$

Then it is clear that there is a point $\mathbf{p} \in \partial B$ such that if one "pushes" the object B at the point $\mathbf{p}$ with a "*nasty finger*" with a force of magnitude only infinitesimally larger than λ, such a finger will be able to break the grasp. Thus $r_{\chi_{nasty}} = \lambda$ defines a grasp metric.

2.5 Grasp Metrics Based on Virtual Coefficients

Yet another formulation of a closure grasp is via *form closure*. Recall that with each nominal point of contact we can also associate a twist system; they describe the degrees of freedom of the body local to that contact point. Thus a system with a set of contacts is free to move by a twist if and only if the virtual coefficient of any wrench and the twist is nonnegative, since otherwise the virtual work done by some wrench would be negative. This situation occurs when the twist $\mathbf{d}$ is *reciprocal* to the bisense wrenches

$$\mathbf{w}_i \odot \mathbf{d} = 0, \quad (\forall i = 1, \ldots, k)$$

and *reciprocal* or *repelling* to the unisense wrenches

$$\mathbf{w}_i \odot \mathbf{d} \geq 0, \quad (\forall i = k+1, \ldots, n).$$

A set of twists (associated with the contacts) is said to constitute a *form closure* if and only if any arbitrary twist is resisted by the set of contacts. That is, the object is *totally constrained* with no degree of freedom left. Thus if $\mathbf{d}$ is an arbitrary twist then it must be *nonreciprocal* to some bisense wrench $\mathbf{w}_i$ $(i = 1, \ldots, k)$ or must be *contrary* to some unisense wrench $\mathbf{w}_i$ $(i = k+1, \ldots, n)$:

$$\begin{aligned} \mathbf{w}_i \odot \mathbf{d} &\neq 0, \quad (\exists i = 1, \ldots, k) \\ \text{or} \quad & \\ \mathbf{w}_i \odot \mathbf{d} &< 0, \quad (\exists i = k+1, \ldots, n). \end{aligned}$$

Put another way, this is equivalent to saying that, for any arbitrary vector $\mathbf{d}' \in \mathbb{R}^6$, we have

$$\begin{aligned} \mathbf{w}_i \cdot \mathbf{d}' &= 0, \quad (\forall i = 1, \ldots, k) \\ \text{implies that} \quad & \\ \mathbf{w}_i \cdot \mathbf{d}' = \pi\mathbf{w}_i \cdot \mathbf{d}' &< 0, \quad (\exists i = k+1, \ldots, n), \end{aligned}$$

which, in turn is equivalent to the condition that

$$\mathbf{0} \in \text{int conv } (\pi\mathbf{w}_{k+1}, \ldots, \pi\mathbf{w}_n)$$

Thus, *force/torque closure* and *form closure* are equivalent.

Also note that, one can use the definition of a form closure to define a grasp metric in terms of the virtual coefficients determined by the system of wrenches of the grasp and a unit twist. For instance, one may propose the following:

1. Let $\mathbf{d}$ be an arbitrary twist, $|\mathbf{d}|_2 = 1$. Then, define the *minimal virtual coefficient* of the grasp $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ with respect to $\mathbf{d}$ to be
$$\mu_{con}(\mathbf{d}) = \max_{i \in I} |\mathbf{w}_i \odot \mathbf{d}|$$
where $i \in I$ if and only if $\mathbf{w}_i$ is contrary to $\mathbf{d}$ $(1 \le i \le n)$, (i.e., $\mathbf{w}_i \odot \mathbf{d} < 0$). Note that $I \neq \emptyset$ if the grasp $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is a closure grasp. Now define the grasp metric to be
$$\begin{aligned} \mu_{con} &= \min\Big(\mu_{con}(\mathbf{d}) : \\ &\qquad \mathbf{d} \in \mathbb{R}^6 \text{ is a unit twist}\Big) \end{aligned}$$
This represents the smallest amount of virtual work an adversary may have to perform to "break the grasp." A little thought will show that this is exactly the grasp metric defined by the residual radius $r_{\chi_{con}}$ of the set $G_{\chi_{con}}$.

2. As earlier, let $\mathbf{d}$ be an arbitrary twist, $|\mathbf{d}|_2 = 1$. Then, define the *minimal virtual coefficient* of the grasp $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ with respect to $\mathbf{d}$ to be
$$\mu_{max}(\mathbf{d}) = \sum_{i \in I} |\mathbf{w}_i \odot \mathbf{d}|$$
where $i \in I$ if and only if $\mathbf{w}_i$ is non-reciprocal to $\mathbf{d}$ $(1 \le i \le k)$ or contrary to $\mathbf{d}$ $(k+1 \le i \le n)$, (i.e., $\mathbf{w}_i \odot \mathbf{d} \neq 0$ or < 0, depending respectively on whether $1 \le i \le k$ or $k+1 \le i \le n$). Now define the grasp metric to be
$$\begin{aligned} \mu_{max} &= \min\Big(\mu_{max}(\mathbf{d}) : \\ &\qquad \mathbf{d} \in \mathbb{R}^6 \text{ is a unit twist}\Big) \end{aligned}$$
One sees that this is exactly the grasp metric defined by the residual radius $r_{\chi_{max}}$ of the set $G_{\chi_{max}}$.

One can also easily devise a grasp metric by means of virtual coefficients that corresponds exactly to one's favorite finger force constraint.

2.6 Some Remarks

We note that most of the grasp metrics considered are dependent on the coordinate system chosen—namely, on the choice of the torque origin. This can be addressed by either asking that the torque origin is always chosen at the *center of mass* of the object, or by considering different measure of the feasible wrench set, e.g., volume. But such solutions seem ad hoc and without an immediate physical interpretation. Another problem is that the torque and force dimensions are not comparable. The scalings chosen in either dimension is clearly artificial, but do affect the grasp metric. A simple solution is to leave the two dimensions separate and define the grasp metrics by a pair of numbers. While we avoid these issues for the time being, we hope to come back to these problems in the future.

2.7 Synthesis of a Grasp

Let us next consider the problem of grasp synthesis with total disregard for the condition of optimality and in the simplest possible situation, where there is no friction at the contact points—the so-called "positive grip."

In order to obtain a particular grasp on an object, it must be determined if that grasp is achievable. It is for this reason, researchers have studied the question of how many fingers (wrenches) are required to obtain certain grasps on the object.

Reuleaux and Somoff determined that the closure grasp of a two dimensional object requires at least *four* wrenches and of a three dimensional object requires at least *seven*, where the wrenches are normal to the surface of the object.

Mishra, Schwartz and Sharir [21] gave general bounds on the number of fingers in the case of a *positive grip*; they also provided an algorithm that finds *at least one* such grip on a polyhedral object and their algorithm runs in time linear in the number of faces of the object. Here, we briefly describe the techniques of Mishra, Schwartz, and Sharir.

Recall that a non-frictional grip is called a *positive grip*. Note that, in this case, the fingers are assumed to be point fingers, a finger can only apply a force on the

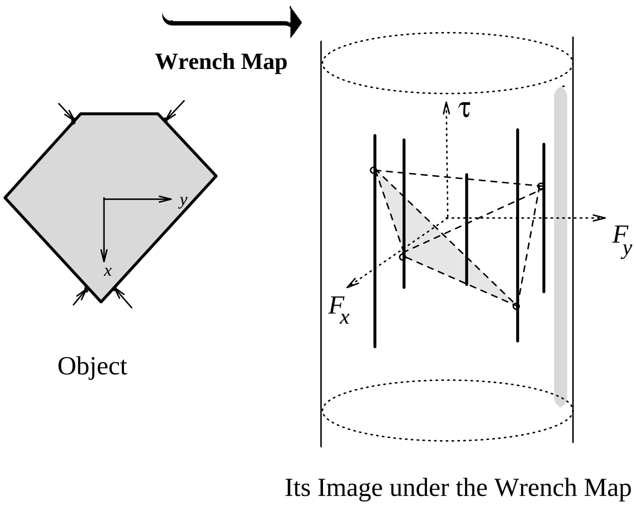

Figure 2: *A pictorial explanation of the techniques of Mishra, Schwartz and Sharir.*

object along the surface-normal at the point of contact, directed inward. In this situation, we have a *wrench map*, Γ, mapping ∂B into the six-dimensional *wrench space* $\mathbb{R}^6$ as follows:

$$\Gamma\colon \partial B \to \mathbb{R}^6 \;: \mathbf{p} \mapsto [\mathbf{n}(\mathbf{p}), \mathbf{p} \times \mathbf{n}(\mathbf{p})].$$

Essentially, Γ maps $\mathbf{p}$ to the point $\Gamma(\mathbf{p})$ ($\in S^2 \oplus \mathbb{R}^3$, a unit radius cylinder) in the wrench space that represents the effects of applying a unit force at $\mathbf{p}$ in the direction $\mathbf{n}(\mathbf{p})$.

Now, it can be shown that

Theorem 1 *Given an arbitrary compact rigid object B whose piece-wise smooth surface ∂B is not a "surface of revolution," B can be held with a closure grasp by a positive grip of at most twelve fingers.* $\square$

The assumption that ∂B is not a surface of revolution is essential, since, in the absence of friction, no matter how many fingers are employed the object can always be rotated about its axis of symmetry. A general description of such exceptional objects is given in [21]. Also, see [29].

By virtue of the discussion of the previous section, we only need to show that

$$\Big(\exists\, \mathbf{p}_1, \ldots, \mathbf{p}_m \in \partial B,\ m \le 12\Big)$$

$$\Big[\mathbf{0} \in \text{int conv}\,\Big(\Gamma(\mathbf{p}_1), \ldots, \Gamma(\mathbf{p}_m)\Big)\Big].$$

The proof is in two steps:

Step 1. $\Rightarrow$ We claim that

$$\mathbf{0} \in \text{int conv}\ \Gamma(\partial B).$$

Using Gauss' divergence theorem, we see that

$$\left[\int_{\partial B} \mathbf{n}(\mathbf{p})\, dS,\ \int_{\partial B} (\mathbf{p} \times \mathbf{n}(\mathbf{p}))\, dS\right] = \mathbf{0},$$

i.e.

$$\mathbf{0} \in \text{conv}\ \Gamma(\partial B).$$

Thus, it remains to be shown that the origin is indeed an interior point of the convex hull; assume to the contrary. Then it can be shown that there is a nonzero vector $\mathbf{g} = [F, \tau] \in \mathbb{R}^6$ orthogonal to the linear subspace spanned by $\Gamma(\partial B)$. Thus, for each function $c(\mathbf{p})$ we have

$$\int_{\partial B} c(\mathbf{p})\ [(F + \tau \times \mathbf{p}) \cdot \mathbf{n}(\mathbf{p})]\ dS = 0.$$

In particular, substitute the function $c(\mathbf{p}) = (F + \tau \times \mathbf{p}) \cdot \mathbf{n}(\mathbf{p})$ into the last equation, to deduce that $(F + \tau \times \mathbf{p}) \cdot \mathbf{n}(\mathbf{p})$ must be identically 0 over ∂B. But, this is possible only in the case when the surface of B is a surface of revolution.

Step 2. $\Rightarrow$ The rest follows by an application of the following theorem from combinatorial geometry:

Theorem 2 (Steinitz's Theorem[30]) *If $X \subseteq \mathbb{R}^d$ and $\mathbf{p} \in$ int conv X, then $\mathbf{p} \in$ int conv Y for some $Y \subseteq X$ with $|Y| \le 2d$.* $\square$

Our main result then follows from Steinitz's Theorem, once we identify X with $\Gamma(\partial B)$, and Y with the set $\{\Gamma(\mathbf{p}_1), \ldots, \Gamma(\mathbf{p}_m)\}$, with $m \le 2 \times 6 = 12$.

Note that here we see how to pick one grasp. There is a simple linear time algorithm to find such a grasp with 12 fingers. An interesting related algorithmic question is to understand the complexity of the problem of choosing an optimal grasp (say, with a fixed number $m \ge 12$ of fingers) such that

$$G_{\chi}(\Gamma(\mathbf{p}_1), \ldots, \Gamma(\mathbf{p}_m))$$

contains as a large a residual ball as possible, under one's favorite faithful, convex and compact finger force constraint. This leads us to the study of the Quantitative Steinitz's Theorem (Q.S.T.).

An argument as above for one- or two-dimensional objects yields a theory and results with appropriate changes in the dimension of the wrench space (one and three, respectively, instead of six) and the number of fingers sufficient for positive closure grasps (two and six, respectively, instead of twelve). Also note that the same line of reasoning leads to a much more tight calculation of number of fingers necessary and sufficient for equilibrium grasps (two for 1-dimensional objects, four for 2-dimensional objects and seven for 3-dimensional objects). Other results for other contact types may be obtained in a similar manner.

3 Q.S.T.

Following the discussion of the earlier sections, we see that the selection of an optimal grasp, with respect to any of the grasp metrics of choice, leads to the study of a stronger *quantitative version* of the Steinitz's theorem. We start with few notations, as follows.

For any convex set $X \subseteq \mathbb{R}^d$, let the *residual ball* of X refer to the maximal ball $\mathcal{B}(X)$ centered at the origin $\mathbf{0}$ such that $\mathcal{B}(X)$ is fully contained inside the convex set X. The *residual radius* of X, denoted $r(X)$ is the radius of this residual ball $B(X)$. By an abuse of notation, we write $r(X)$ instead of $r(\text{conv } X)$, if X is not convex. Let

$$\begin{aligned} r_d(m, X) &= \max\{r(Y): \\ &\qquad Y \subseteq X \text{ and } |Y| \leq m\}, \end{aligned}$$

$$\begin{aligned} r_d(m) &= \min\{r_d(m, X): \\ &\qquad X \subseteq \mathbb{R}^d \text{ and } r(X) \geq 1\}. \end{aligned}$$

In the notation, we shall omit the subscript d, if the dimension is clear from the context. (In this paper, the interesting case is $d = 6$.) Thus the original Steinitz's theorem can now be interpreted to say that

$$r_d(2d) > 0.$$

A quantitative version of Steinitz's theorem provides more precise bounds for the number $r_d(m)$, when $m \geq 2d$.

Now the optimal closure grasp with m fingers can be expressed in terms of the residual radius values given by a quantitative Steinitz's theorem. For the sake of simplicity consider the finger force constraint given by χ_{con}. Then the grasp metric for an optimal closure grasp with an m-fingered positive grip for a body B can be seen to be the value, $r_6(m, \text{conv } \Gamma(\partial B))$. To see this, note that if we choose m points in $G_{\chi_{con}} = \text{conv } \Gamma(\partial B)$ with residual radius r then any external wrench vector $\mathbf{v}$ of magnitude at most r can be written as a convex combination of the m chosen points. So if $\mathbf{v}$ is any external wrench that is applied to the body B, and $\mathbf{v}$ lies in the residual ball of radius r, we can resist this external wrench by applying suitable forces (of magnitude at most 1) at the grasp points such that these forces sum to $-\mathbf{v}$; hence, we maintain the body in equilibrium. Thus, we see that the quantity $r_6(m)$ gives a universal measure for the quality of a closure grasp with $m \geq 12$ fingers.

3.1 B.K.P. Bounds

The special case, where $m = 2d$ had been studied by Bárány, Katchalski and Pach [2]; they showed that

$$r_d(2d) > \frac{c}{(2ed)^{\lfloor d/2 \rfloor} d^2}.$$

These results seem to indicate that a twelve-finger positive grip, while sufficient to provide a closure grasp, may not be adequate to achieve a desirable grasp quality.

Theorem 3 (Q.S.T.)
Bárány, Katchalski and Pach [2]
For any positive d there is a constant $r = r_d(2d) > d^{-2d}$ such that given any set $X \subseteq \mathbb{R}^d$ of points in d-space whose convex hull contains the unit ball $\mathcal{B}^d$ centered at the origin $\mathbf{0}$, there is a subset $Y \subseteq X$ with at most $2d$ points whose convex hull contains a ball centered at $\mathbf{0}$ with radius r.

Proof

The proof is constructive. We first choose $(d+1)$ rays, placed regularly as follows: Let Δ be a regular d-simplex inscribed in the unit ball $\mathcal{B}$. By assumption, Δ is contained in the conv X. The desired rays R_i $(1 \le i \le d+1)$ are the ones joining the origin to the vertices of Δ. Let $p_i' = R_i \cap \partial\text{conv } X$. Each such p_i' lies on a face of the conv X and thus can be expressed as a convex combination of at most d points of X. The totality of these $d(d+1)$ points $Y' \subseteq X$ contains Δ and thus a ball of radius $1/d$ in its convex hull. Let $P = \text{conv } Y'$ and $P_1, P_2, \ldots, P_l$ be the facets of P, each of which may be assumed to a simplex (otherwise triangulate nonsimplicial facets). Clearly, $l \le \binom{d(d+1)}{d}$. Also note that pos P_i's cover the sphere S^{d-1}. Choose a facet, say P_1, such that the surface area of $S \cap \text{pos } P_1$ is as large as possible and thus greater than

$$\frac{\int_{S^{d-1}} dA}{l}.$$

Now choose two rays $R_1' \in \text{pos } P_1$ and $R_2' = -R_1'$ such that the minimal angle between R_1' and the facets of P_1 is maximal and equal to α. Again, choose d points of X such that their convex hull contains the point $R_2' \cap \partial\text{conv } X$. These points and the vertices of P_1 is the desired Y. Clearly $|Y| \le 2d$ and has residual radius r, where

$$\begin{aligned} r &= \frac{\sin\alpha}{\alpha}\left(1 + \frac{1}{d^2} + \frac{2\cos\alpha}{d}\right)^{1/2} \\ \alpha &> \arctan \frac{\int_{S^{d-1}} dA}{l d \int_{S^{d-1}} dA}. \end{aligned}$$

Simple calculation then shows that $r_d(2d) > d^{-2d}$. Additional efforts lead to the improvement mentioned earlier and uses Upper Bound Theorem for a tighter estimation of the number of facets of P. $\square$

3.2 K.M.Y. Bounds

Kirkpatrick, Mishra and Yap [14] have provided more general bounds. Here, we consider the d-dimensional case for $d > 2$. The techniques are slightly weaker than the 2-dimensional case considered in greater entails in [14].

3.2.1 Lower Bound

We first give a lower bound for $r_d(m)$ for sufficiently large m (in particular, for all $m \ge 13^d d^{\frac{d+3}{2}}$). Thus, m is chosen to be large enough to guarantee that

$$k = \left\lfloor \left(\frac{m}{2d^2}\right)^{1/(d-1)} \right\rfloor$$

takes integral values, greater than $\lceil 11\sqrt{d} \rceil$.

Lemma 1 *For any set $X \subseteq \mathrm{I\!R}^d$ whose convex hull contains the unit ball $\mathcal{B}^d$ centered at the origin $\mathbf{0}$, we can find a set $Y \subseteq X$ of at most m points with residual radius*

$$r(Y) \ge 1 - 3d\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}}, \quad \textit{for all } m \ge 13^d d^{\frac{d+3}{2}}.$$

Proof

Let k be defined as a function of d and m, as before. It suffices to show that

$$r(Y) \geq 1 - \frac{97}{48}\frac{d}{(k-1)^2} > 1 - \frac{3d}{(k+1)^2},$$

in the given range for k.

Henceforth, P will stand for the convex hull of X. Let C be the d-dimensional cube whose faces are normal to the appropriate coordinate axes, of side-length 2 and containing the unit ball $\mathcal{B}^d$. On each face of C we place a $k \times k \times \cdots \times k$ ($d-1$ times) grid (so the grid points have coordinates that are integer multiples of $\frac{1}{k-1}$ and two adjacent grid points are $\frac{2}{k-1}$ apart). Note that there are fewer than $2dk^{d-1} \leq m/d$ 'grid cubes' on the union of the $2d$ faces of C. Through each grid point p, we pass a ray R from the origin. Let R intersect the unit sphere S^{d-1} at $y(R)$. For each such ray R, we choose at most d vertices of P (the convex hull of X) as follows. If the ray passes through an i-face of P, we choose $i+1$ vertices of P whose convex span intersects that ray and is contained in that i-face. Thus the set Y of chosen vertices has at most m points. The convex hull of Y contains the set Y' of all points of the form $y(R)$ where R is a ray passing through the grid point.

Let R be any ray originating from $o = \mathbf{0}$ and suppose it intersects some face of C at a point a where a lies inside a grid cube S. Consider the triangle oab where b is any other point on the boundary of S.

$$\begin{aligned} \sin\angle(aob) &= |ab| \cdot \frac{\sin\angle(oab)}{|ob|} \\ &\leq \frac{2\sqrt{d}}{k-1} \cdot \frac{1}{1} \leq \frac{1}{5} \end{aligned}$$

Choose α to be

$$\alpha = \arcsin \frac{2\sqrt{d}}{k-1}.$$

Let q_0 be any point at distance $\cos\alpha$ from the origin. We show that q_0 lies in the convex hull of Y'. Let R_0 be the ray from $o = \mathbf{0}$ through q_0 and suppose R_0 intersects the grid cube S_0. Let K_0 be the cone bounded by the set of rays originating from o that makes an angle of α with R_0. Hence each ray that passes through a vertex of S_0 is contained in K_0. There is a unique hyperplane H_0 containing $\partial(K_0) \cap S^{d-1}$. Note that $q_0 = R_0 \cap H_0$. Let

$$T_0 = \{y(R) : R \text{ passes through a vertex of } S_0\}$$

and

$$T_1 = \{R \cap H_0 : R \text{ passes through a vertex of } S_0\}.$$

By definition, $T_0 \subseteq Y'$. Note that each point in T_0 lies on the side of H_0 not containing the origin. This means that the convex hull of Y' contains the set T_1. But the convex span of the set T_1 contains the point $q_0 = R_0 \cap H_0$. This proves $r(Y) \geq r(Y') \geq \cos\alpha$.

$$\begin{aligned} \cos\alpha &= (1 - \sin^2\alpha)^{1/2} \\ &> 1 - \frac{\sin^2\alpha}{2} - \frac{\sin^4\alpha}{8}\sum_{i=0}^{\infty} \sin^{2i}\alpha \\ &= 1 - \frac{\sin^2\alpha}{2} - \frac{\sin^2\alpha}{8}\left[\frac{\sin^2\alpha}{1-\sin^2\alpha}\right] \\ &\geq 1 - \frac{97\sin^2\alpha}{192} \\ &\qquad (\text{since } \sin^2\alpha \leq 1/25) \\ &\geq 1 - \frac{97d}{48(k-1)^2}. \end{aligned}$$

This proves the lower bound lemma. □

3.2.2 Upper Bound

Next, we derive an upper bound for $r_d(m)$. For this purpose, we let X be all the points on the unit sphere and then bound the largest radius of a ball contained in the convex hull of m points on the unit sphere. The convex hull of any such m points forms a polytope. The proof relies on the facts that (1) any "long" edges of this polytope bound the radius of the contained ball and (2) since the polytope has only m vertices it must have some "long" edges. The detailed calculations provide an appropriate numerical bound.

Lemma 2 *Let $X \subseteq \mathbb{R}^d$ be the set of all points on the surface of the d-dimensional unit ball centered at the* *origin* $\mathbf{0}$*. Thus, the convex hull of* X *contains the unit ball* $\mathcal{B}^d$ *centered at the origin* $\mathbf{0}$*. Then any set* $Y \subseteq X$ *of at most* m *points has a residual radius*

$$r(Y) \leq 1 - \frac{1}{17}\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}}, \quad \textit{for all } m \geq 3^d d^2.$$

Proof

The proof proceeds in two steps: We first show that for all $m > 0$ and for all $0 < \Theta < \pi/4$,

$$r(Y) \leq \max\left(\cos\frac{\Theta}{2}, 1 - \frac{1-\tan^2\Theta}{16}\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}}\right).$$

Then by an appropriate choice of the parameter Θ ($\Theta = 4\pi/53$), we obtain the claimed bound.

(1) Let Y be a set of m points in $\mathbb{R}^d$ all lying on the surface of a unit ball and $P = \text{ConvexHull}(Y)$. Let P' be the polyhedron obtained from P by triangulating the nonsimplicial facets of P. Let pq be an edge of the polyhedron P'. Then

$$r(Y) \leq \cos\frac{\angle(poq)}{2}.$$

Thus, if

$$\alpha = \max_{pq = \text{edge of } P'} \angle(poq),$$

is the maximum of all such angles, then

$$r(Y) \leq \cos\frac{\alpha}{2}.$$

If $\alpha \geq \Theta$ then

$$r(Y) \leq \cos\frac{\Theta}{2}.$$

Henceforth, we assume that $\alpha < \Theta$. Let t stand for $\tan\Theta$; thus $0 < t < 1$.

(2) Let $p \in Y$ be any point, and define its *truncated cone* K_p as follows:

$$K_p = \{x \colon \angle(xop) \leq \alpha \text{ and } x \cdot p \leq 1\}.$$

Now, if q is an arbitrary point on the surface of the unit ball, then the line segment oq belongs to K_p, for each vertex p of some (simplicial) facet of P'. As each such simplex facet has d vertices, the collection of truncated cones cover each point in the unit ball at least d times. Thus, we see that

$$m \cdot \text{Volume}(K_p) \geq d \cdot \text{Volume(unit ball)}.$$

Let $V_d(r)$ stand for the volume of a d-dimensional ball of radius r.

$$V_d(r) = V_d(1) r^d.$$

Thus, the volume of the d-dimensional unit ball is given by

$$\begin{aligned} V_d(1) &= 2\int_0^{\pi/2} V_{d-1}(\sin\theta)\ \sin\theta\ d\theta \\ &= 2V_{d-1}(1)\int_0^{\pi/2} \sin^d\theta\ d\theta \\ &= K(d)V_{d-1}(1), \end{aligned}$$

where $K(d)$ is defined by the last equation. The volume of each K_p is given by

$$\begin{aligned} \text{Volume}(K_p) &= \int_0^1 V_{d-1}(r\tan\alpha)\ dr \\ &= V_{d-1}(\tan\alpha)\int_0^1 r^{d-1}\ dr \\ &= \frac{V_{d-1}(\tan\alpha)}{d}. \end{aligned}$$

Substituting the volumes into the preceding inequality, we get

$$m\frac{\tan^{d-1}\alpha V_{d-1}(1)}{d} \geq dK(d)V_{d-1}(1).$$

Hence,

$$\begin{aligned} 1 &> t = \tan\Theta > \tan\alpha \\ &\geq \left(\frac{d^2K(d)}{m}\right)^{\frac{1}{d-1}} = c(d,m), \end{aligned}$$

where $c(d,m)$ is defined in the last equation. Using the inequality $c(d,m)^2 < t^2$, we get

$$\begin{aligned} \cos^2\alpha &\leq \frac{1}{1+c(d,m)^2} \\ &\leq 1 - c(d,m)^2 + c(d,m)^4 \\ &\leq 1 - (1-t^2)c(d,m)^2. \end{aligned}$$

Hence,

$$\begin{aligned}\cos\alpha &= 2\cos^2\frac{\alpha}{2}-1\\ &\le \left(1-(1-t^2)c(d,m)^2\right)^{\frac{1}{2}}\\ &\le 1-\frac{1-t^2}{2}c(d,m)^2\end{aligned}$$

and

$$\cos^2\frac{\alpha}{2}\le 1-\frac{1-t^2}{4}c(d,m)^2.$$

Finally, we get

$$\begin{aligned}\cos\frac{\alpha}{2} &\le \left(1-\frac{1-t^2}{4}c(d,m)^2\right)^{\frac{1}{2}}\\ &\le 1-\frac{1-t^2}{8}c(d,m)^2.\end{aligned}$$

Hence,

$$r(Y)\le 1-\frac{1-t^2}{8}\left(\frac{d^2K(d)}{m}\right)^{\frac{2}{d-1}}.$$

(3) Note that (e.g., [11],page 369)

$$\begin{aligned}K(d) &= 2\int_0^{\pi/2}\sin^d\theta\, d\theta\\ &= \begin{cases}\frac{(2k-1)!!}{(2k)!!}\pi, & \text{if } d=2k=\text{even};\\ \frac{(2k)!!}{(2k+1)!!}2, & \text{if } d=2k+1=\text{odd}.\end{cases}\\ &\ge 2\left(\frac{1}{2}\right)^{\frac{d-1}{2}}.\end{aligned}$$

Here $k!!$ stands for $k(k-2)(k-4)\cdots(\ell+4)(\ell+2)\ell$ (terminating in $\ell=1$ or 2, depending on whether k is odd or even). Thus

$$r(Y)\le 1-\frac{1-\tan^2\Theta}{16}\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}}.$$

(4) The stated bound follows with appropriate choice of the parameter Θ, as shown below: Let $m\ge 3^d d^2$; then

$$\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}}<\frac{1}{9}.$$

Choose the parameter $\Theta=4\pi/53$, and observe that

$$\cos\frac{2\pi}{53}<1-\frac{1}{17\times 9}\le 1-\frac{1}{17}\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}}.$$

Since $1-\tan^2\frac{4\pi}{53}>16/17$,

$$\begin{aligned}&1-\frac{1-\tan^2(4\pi/53)}{16}\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}}\\ &\le 1-\frac{1}{17}\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}}.\end{aligned}$$

□

If we choose $\Theta=\pi/5$ in the preceding proof, we can show that: for all $m>0$,

$$r(Y)\le 1-\frac{15}{512}\left(\frac{d}{m}\right)^{\frac{2}{d-1}}.$$

If $m\le d$, then

$$0=r(Y)<1-\frac{15}{512}\left(\frac{d}{m}\right)^{\frac{2}{d-1}}.$$

On the other hand, if $m>d$, we get the result since $\cos\frac{\pi}{10}<1-\frac{15}{512}$ and $1-\tan^2\frac{\pi}{5}>15/32$.

Summarizing the preceding lemmas, we have

Theorem 4 *For all $m\ge 13^d d^{\frac{d+3}{2}}$,*

$$\frac{1}{17}\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}}\le 1-r_d(m)\le 3d\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}}.$$

□

The results in this section seem highly pessimistic for moderately small number of fingers. However, if one allows large number of fingers, or allows frictional and/or soft contact models, there is a possibility for synthesizing moderately efficient closure grasps. Kirkpatrick et. al. [14] have provided certain approximation algorithms for these problems and some related computational geometric problems. However, still much research is needed to provide practical algorithms.

4 Nasty Finger Model

Here, we present a result due to Walter Meyer ("A Seven Finger Robot Hand is Weak," unpublished manuscript, Adelphi University), which shows that there is a family of boxes (rectangular parallelo pipeds), "skinny boxes," all of unit length, where the optimal grasp (under r_{nasty} model) has a grasp metric value bounded by a linear function of the skinny dimension. These results have been motivated by the model proposed by Kirkpatrick, Mishra and Yap [14] who left the problem for the small number (≥ 7) of fingers unanswered. Note that the problem of quantifying the trade-offs between the number of fingers and the best achievable grasp metric values remain largely open in the most general context.

Observe that as the dimension of the object to be grasped become smaller, the ability of a finger to generate large torques become weaker. However, it is also equally harder for the external ("hostile" or "nasty") environment to impose a large counteracting torque to break a grasp on a small body. That is, the robot hand and its opponent (adversarial environment) are equally balanced. This balance is made explicit, in our discussion, by always assuming that the object, B, to be grasped is scaled in such a manner that $R_{\chi}(\partial B)$ contains a unit residual sphere in its interior. That is, if we are allowed to use arbitrarily large number of fingers (in the limit going to infinity) then we can resist any external wrench of unit magnitude. (Note, however, for this argument to work out, one has to rule out such finger force constraints as χ_{max} and replace it by an appropriate version of χ_{hyb} with some bound on the number of the partitions. With χ_{max}, as the number of fingers increase so does the bound on the volume of the resistable wrench set.)

Now consider a family of rectangular parallelopipeds—"skinny boxes"—each of which has one side of unit length and the remaining two sides equal and of length $\varepsilon < 1$. We denote such a box by Box_ε. Then the following result holds:

Theorem 5 (Meyer) *For any seven fingered positive grip of a skinny box,* Box_ε

$$r_{nasty} \leq 5\sqrt{2}\varepsilon.$$

Proof

Note that any closure grasp of the object must place at least one finger on each of the six faces of the box.

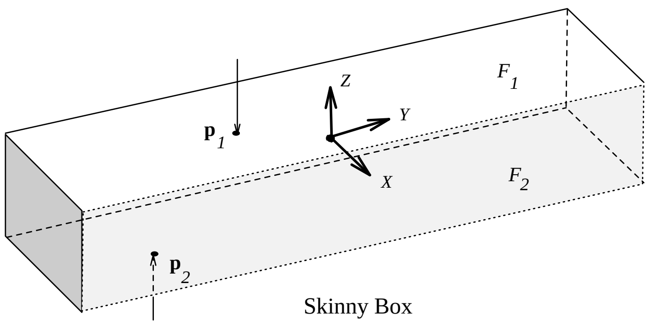

Figure 3: *The "skinny box" example.*

Thus there can be exactly one face with two fingers. Thus there are at least two opposing parallel long faces F_1 and F_2 with one finger on each. The contact points are $\mathbf{p}_1 \in F_1$ and $\mathbf{p}_2 \in F_2$. Let us now assign a coordinated system with the origin an the center of mass of the skinny box, the Z axis normal to the faces F_1 and F_2 and the X and Y axes normal to the other four faces. That is, X-Y plane is parallel to the faces F_1 and F_2. Note that the torques generated by the contacts $\mathbf{p}_1$ and $\mathbf{p}_2$ are in the X-Y plane. Because of the choice of the dimensions of the skinny box, the torque component due to the other five points of contact in either the X or Y direction is no more than $\varepsilon/2$; thus, making the total torque component in the X-Y plane due to these five fingers no more than $5\sqrt{2}\varepsilon/2$.

Now assume that the magnitude of the forces at the points of contact $\mathbf{p}_1$ and $\mathbf{p}_2$ are respectively f_1 and f_2. Let us now place the "nasty finger" close to one of the corners of the faces F_1 and F_2 at a point $\mathbf{p}_{nasty}$, exerting a force of magnitude λ. Without loss of generality, assume that $\mathbf{p}_{nasty} \in F_1$. Then, by the force and torque equilibria conditions, we have

$$\begin{aligned} |(f_1\mathbf{p}_1 - f_2\mathbf{p}_2 + \lambda\mathbf{p}_{nasty}) \times Z| &\leq 5\sqrt{2}\varepsilon/2 \\ \lambda &= |f_1 - f_2| \end{aligned}$$

Since

$$\max(|\mathbf{p}_1 - \mathbf{p}_{nasty}|, |\mathbf{p}_2 - \mathbf{p}_{nasty}|) \geq 1/2,$$

by design, we see that $\lambda \leq 5\sqrt{2}\varepsilon$. □

Note however, that the argument above fails for nine fingers. That is with nine fingers, one can devise a grasp (for every pair of opposing faces, place two fingers on a face and one on the opposing face) that has a good value for this grasp metric, independent of the skinny dimension of the box. Also, note that this argument fails to generalize to the other grasp metrics directly. These issues need to be explored further.

5 Optimal Three-finger Planar Grasps

For the sake of concreteness, we now consider some special cases, where we wish to obtain the best three-finger grasp of a planar polygonal object assuming non-frictional contacts. Note that in this case, since it is not possible to guarantee that the resulting grasp will have the force/closure properties, we are willing to sacrifice the condition requiring torque-closure. In other words, we wish only to achieve a three-finger grasp such that the smallest external force such a grasp and resist is as large as possible.

More formally given a simple n-gon P, we wish to choose three distinct points $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$ on the interior of the edge segments of P such that the following properties hold:

1. The unit inner normals $\mathbf{n}(\mathbf{p}_1)$, $\mathbf{n}(\mathbf{p}_2)$ and $\mathbf{n}(\mathbf{p}_3)$ are concurrent.

2. The unit inner normals $\mathbf{n}(\mathbf{p}_1)$, $\mathbf{n}(\mathbf{p}_2)$ and $\mathbf{n}(\mathbf{p}_3)$ positively spans the two-dimensional force space, i.e.,
$$(\forall \mathbf{w} \in \mathbb{R}^2)\ (\exists f_i \geq 0, 1 \leq i \leq 3)\ \mathbf{w} = \sum_{i=1}^{3} f_i \mathbf{n}(\mathbf{p}_i).$$

3. The unit normals are "well-balanced" in the sense that
$$\min\Big\{|\mathbf{w}| : \mathbf{w} \in \mathbb{R}^2,\ (\exists f_i \geq 0, 1 \leq i \leq 3)\ \chi(f_1, f_2, f_3) = 1,\ \mathbf{w} = \sum_{i=1}^{3} f_i \mathbf{n}(\mathbf{p}_i)\Big\},$$
is as large as possible (among all choices of $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$). Here, $\chi(f_1, f_2, f_3)$ denotes a finger force constraint condition on the magnitude of the forces applied at the points of contact. For instance,
$$\chi_{con} : f_i \geq 0, \sum f_i \leq 1,$$
or
$$\chi_{max} : f_i \geq 0, \max f_i \leq 1.$$

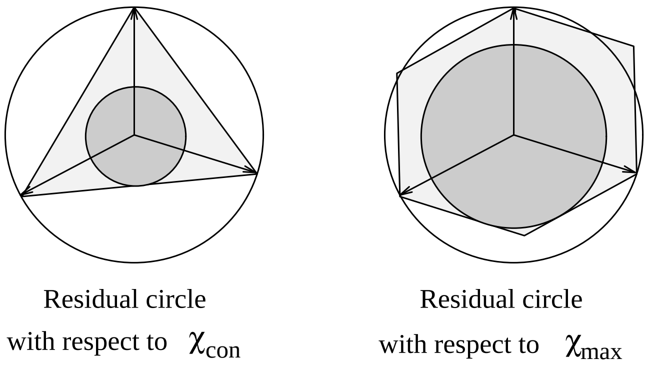

Figure 4: *Grasp metrics associated with χ_{con} and χ_{max}.*

Thus the first property denotes the trivial torque equilibrium condition; the second property denotes the force closure condition and the third property measures the goodness of the grasp. In English, the third property says: under the condition χ_{con}, we wish to maximize the radius of a disk, centered at origin and contained in the triangle formed by (convex hull of) the points (on the unit circle) corresponding to the vectors $\mathbf{n}(\mathbf{p}_1)$, $\mathbf{n}(\mathbf{p}_2)$ and $\mathbf{n}(\mathbf{p}_3)$. Similarly, under the condition χ_{max}, we wish to maximize the radius of a disk, centered at origin and contained in the Minkowski sum of the points (on the unit circle) corresponding to the vectors $\mathbf{n}(\mathbf{p}_1)$, $\mathbf{n}(\mathbf{p}_2)$ and $\mathbf{n}(\mathbf{p}_3)$—a convex hexagon.

Let the corresponding radii be denoted as $\rho_{con}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ and $\rho_{max}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$, respectively. Note that, if the angle α_i's $(1 \leq i \leq 3)$ denote the angles between the inner normals then $\alpha_{max} = \max(\alpha_1, \alpha_2, \alpha_3) \geq 2\pi/3$ completely determines the radii
$$\begin{aligned} \rho_{con} &= \cos(\alpha_{max}/2), \quad \text{and} \\ \rho_{max} &= \sin \alpha_{max}. \end{aligned}$$

Thus both these metrics are monotonically decreasing functions of $2\pi/3 \leq \alpha_{max} \leq \pi$, and it suffices to minimize α_{max}. However, for the sake of the ease of exposition, we will frequently use $\rho = \rho_{con}$, and refer to it as the "*residual radius*" of $\mathbf{n}(\mathbf{p}_1)$, $\mathbf{n}(\mathbf{p}_2)$ and $\mathbf{n}(\mathbf{p}_3)$. The optimal value of residual radius is denoted by ρ^*.

Note that given an edge $e = ab$ of the polygon P, for every point $p \in ab$, $\mathbf{n}(\mathbf{p})$ defines a unique point $q(e)$ on the unit circle in $\mathbb{R}^2$. Thus we may simply refer to this point on the unit circle by $q(e)$. Henceforth, let the edges of the n-gon be given as $E = \{e_1, e_2, \ldots, e_n\}$ and the corresponding points on the unit circle be $Q = \{q_1, q_2, \ldots, q_n\}$, where $q_i = q(e_i)$ $(1 \le i \le n)$.

We may note at this point that there is a trivial $O(n^3)$ time algorithm to find an optimal grasp of a simple n-gon, P, by exhaustively enumerating all edge triples of P ad by examining each triple successively. In order for an edge triple (e_i, e_j, e_k) to produce three necessary optimal contact points, it must be the case that (q_i, q_j, q_k) form a triangle with a positive residual radius of ρ^*—a condition that can be checked easily in $O(1)$ time. However, this is not sufficient—since we must check that there are three points $\mathbf{p}_i \in e_i$, $\mathbf{p}_j \in e_j$ and $\mathbf{p}_k \in e_k$ satisfying the torque equilibrium condition; namely, that $\mathbf{n}(\mathbf{p}_1)$, $\mathbf{n}(\mathbf{p}_2)$ and $\mathbf{n}(\mathbf{p}_3)$ are concurrent meeting at some point c.

This is not hard but requires some thought. We proceed as follows: Consider an edge ab of P. Let $HP(a, ab)$ be the open half plane containing ab and delimited by a line containing a and normal to ab and similarly, let $HP(b, ab)$ be the open half plane containing ab and delimited by a line containing b and normal to ab. Let

$$\mathrm{slab}(e) = HP(a, ab) \cap HP(b, ab),$$

where $e = ab$.

Then it is easy to see that for a triple of edges (e_i, e_j, e_k) to satisfy the torque equilibrium condition, it is necessary and sufficient that

$$\mathrm{slab}(e_i) \cap \mathrm{slab}(e_j) \cap \mathrm{slab}(e_k) = C \neq \emptyset.$$

The point of concurrency $c \in S$, and the contact points $\mathbf{p}_i$, $\mathbf{p}_j$ and $\mathbf{p}_k$ are determined by the normals from c onto the edges e_i, e_j and e_k.

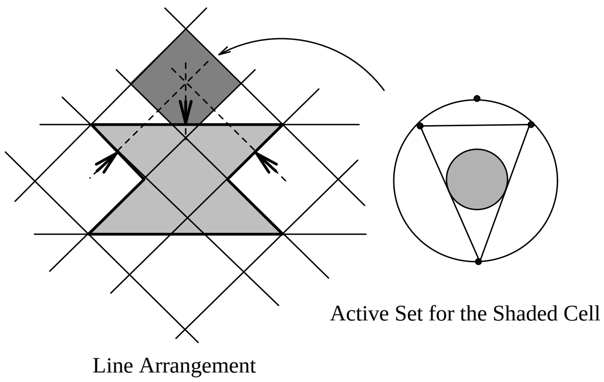

Figure 5: *The line arrangement associated with an object.*

Thus our previous arguments can be summarized to be saying that an edge triple (e_i, e_j, e_k) defines an optimal grasp if $\mathrm{slab}(e_i) \cap \mathrm{slab}(e_j) \cap \mathrm{slab}(e_k)$ is nonempty and that the triangle formed by the corresponding points on the unit circle has a positive residual radius of ρ^*, maximal among all choices of edge triples. These considerations yield an $O(n^3)$-time algorithm.

5.1 An Improved Algorithm

Next, we ask if it is possible to improve upon the trivial $O(n^3)$-time algorithm. Here, we present an $O(n^2 \log n)$-time algorithm for finding the optimal three fingered planar grasp for an arbitrary simple polygon and some simple improvements for convex polygon. We first describe the algorithm assuming that the polygon P is nondegenerate (in the sense that will be made precise later) and then remark on how the nondegeneracy can be eliminated by a simple modification to the algorithm.

The algorithm can be described as follows: First we create the two-dimensional line arrangement formed by a collection of lines consisting of three lines per edge, where the triplet of lines associated with an edge ab are: (1) the line containing the edge ab, (2) the line normal to ab, containing a and (3) the line normal to ab, containing b. Now consider a nonempty cell C of this arrangement: we say a point $q = q(e)$ on the unit circle is *active* for this cell, if $\mathrm{slab}(e) \supseteq C$. The subset of points on the unit circle (among the points $q_1, q_2, \ldots, q_n$ of Q) that are active for this cell C, is called its active set and denoted by $\mathrm{active}(C) \subseteq Q$. Now, if we find three points q_i, q_j and $q_k \in$ active(C), whose residual radius $\rho(C)$ is as large as possible (and positive), then it is seen that ρ^* is simply the maximum of all $\rho(C)$'s taken over all cells of the arrangement.

Note that there are at most $O(n^2)$ cells altogether and as we go from one cell C to its adjacent cell C' then the active(C') can be computed from the active(C) by adding or deleting a point on the unit circle, depending on the line containing the $C \cap C'$. Of course, here we have tacitly assumed that the polygon is nondegenerate, in the sense that the all the lines on the arrangement are distinct, since otherwise $C \cap C'$ may belong to more than one line of the arrangement and thus require addition and deletion of more than one point of the set Q. Clearly, the active sets for all the cells can be computed in $O(n^2)$ time by visiting the cells of the arrangement, starting from a cell with an empty active set (such a cell exists sufficiently far away from the polygon P). However, computing the $\rho(C)$ for each cell may still take $O(n)$ time, thus forcing the entire procedure to take $O(n^3)$ time.

We circumvent this problem by the following simple trick: First of all we maintain the active(C)'s in a clockwise order in a dynamic balanced binary search tree. Since each update operation on this data structure takes $O(\log n)$ time, this increases the complexity of computing the active sets of all the cells to $O(n^2 \log n)$-time.

At any instant, we only remember $\tilde{\rho}$—the maximal residual radius seen so far. That is, $\tilde{\rho}$ is simply the maximum of those $\rho(C)$'s corresponding to only those cells C that have been visited so far. We also remember the edge triple associated with the radius value $\tilde{\rho}$. When we go from a visited cell C to an adjacent unvisited cell C', we do one of two things: If going to the next cell entails deletion of a point, q_i, on the unit circle, then we only have to update the active(C'); the maximal residual radius of C' cannot be larger than that of C and thus $\tilde{\rho}$ remains unchanged. If going to the next cell, on the other hand, entails addition of a point, q_i on the unit circle, then we have to both update the active(C') and check if $\tilde{\rho}$ can be improved. If the maximal residual radius of C', $\rho(C') > \tilde{\rho}$, then the associated triplet from active(C') must involve the new

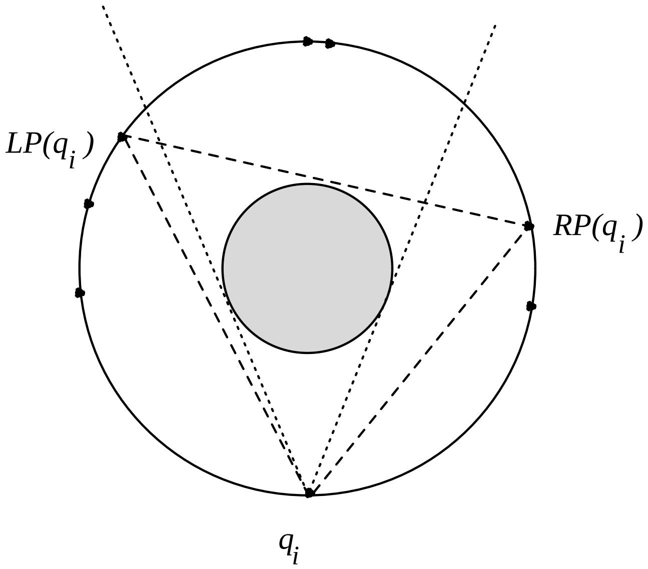

Figure 6: *Test involving q_i and a possible residual radius value of ρ_k.*

point q_i and two of the old points. How can we do this operation quickly?

First note that residual radii cannot take all possible values but only one of $\binom{n}{2}$ values, each value being determined by a pair of distinct points q_l and q_m and is equal to the radius of the circle that is centered at the origin and has the line containing q_l and q_m as tangent. All these radii can be sorted in $O(n^2 \log n)$ time and are denoted by

$$0 \le \rho_1 \le \rho_2 \le \cdots \le \rho_i \le \cdots < 1$$

Suppose before visiting the cell C' the maximal residual radius seen so far is $\tilde{\rho} = \rho_i$. When we go to the cell C' (which requires adding the point q_i), we will successively test if it has a residual radius no smaller than ρ_{i+1}, ρ_{i+2}, etc. until we fail for some value ρ_j $(j > i)$. Each such test can be performed in $O(\log n)$ time as explained below.

Let $i < k \le j$, and we wish to test if active(C') has three points involving q_i and of residual radius $\ge \rho_k$. Consider a circle $C(\rho_k)$ of radius ρ_k and centered at the origin. Two distinct points of active(C') are said to be *mutually visible* if the line segment connecting these two points do not intersect the interior of $C(\rho_k)$. Thus our test succeeds if we can find a pair of mutually visible distinct points among the active(C'), each of which is also mutually visible with q_i. Let the *leftmost partner* of q_i be the *last* mutually visible point of q_i encountered, visiting the points of active(C') in clockwise order starting from q_i. We call this point $LP(q_i)$. Similarly, we define the the *rightmost partner* of q_i by visiting the points of active(C') in anti-clockwise order, and call it $RP(q_i)$. Since the active points of C' are kept in their sorted order in a balanced search structure, both $LP(q_i)$ and $RP(q_i)$ can be computed in $O(\log n)$ time. Then it only remains to check that $LP(q_i)$ and $RP(q_i)$ are mutually visible, a step that can be accomplished in $O(1)$ time.

Thus, we can keep track of $\tilde{\rho}$ by performing a sequence of tests per each new cell, each of which takes $O(\log n)$ time. Note that while there is no a priori bound on the number of tests we may need to perform for a new cell, it should be obvious that all but the last test succeeds and the last test fails. Thus there are at most one test per cell that fails, and the totality of all such failed tests incur a cost of $O(n^2 \log n)$. On the other hand, if we have a successful test involving a radius value ρ_k, then *we shall never perform another successful test involving ρ_k, subsequently.* Thus, the total number of successful tests are bounded by the number possible radii values ($\binom{n}{2}$ of those) and altogether they incur a cost of $O(n^2 \log n)$. Clearly, when we are done visiting all the cells, we have the global maximal residual radius ρ^* together with the edge triple, which readily give the three contact points, and we have spent $O(n^2 \log n)$ time.

If the polygon P is degenerate then the resulting arrangement may force us to add and delete many points of Q while going from a cell to its adjacent cell. If we enforce the discipline that all the deletions are performed before all the additions and each update is performed sequentially then the correctness of the algorithm still holds and the performance analysis goes through *mutatis utandis*. In summary, we have

Theorem 6 *Given an arbitrary simple n-gon P, in $O(n^2 \log n)$ time, we can compute a three finger optimal grasp of P.*

Proof

The complexity analysis follows from the discussion preceding the theorem: The possible radii values can be computed and sorted in $O(n^2 \log n)$ time; the cells can all be visited with the active sets computation taking $O(n^2 \log n)$ time; the tests involved in going from cell to cell are no more than the sum of the possible radii values and the number of cells in the arrangement with each test taking $O(\log n)$ time and thus contributing only $O(n^2 \log n)$ cost to the total cost.

To see the correctness of the algorithm, note that if the computed value at the end is $\tilde{\rho}$ then clearly

$$\tilde{\rho} \leq \rho^*.$$

Conversely, consider the set of edge triples that lead to the maximal residual radius ρ^* and all the cells that are contained in the intersection of three slabs associated with each such edge triple. Among all such cells consider the one that was visited the earliest, say C'. Let the preceding visited cell be denoted C. Let the maximal residual value seen up to the time C was visited be $\tilde{\rho}'$. Thus $\tilde{\rho}' \leq \rho(C) < \rho^* = \rho(C')$. Thus active$(C')$ must have been obtained by addition of a point $q_i \in Q$. Thus, active(C') has two other points q_j and q_k such that a residual circle of radius ρ^* touches an edge of the triangle formed by q_i, q_j and q_k. Thus ρ^* is a possible residual radius and the tests at the cell C' involving possible residual radii values larger than $\tilde{\rho}'$ will all succeed up to ρ^*. Thus if the computed value at the end is $\tilde{\rho}$ then

$$\tilde{\rho} \geq \rho^*.$$

Hence they must be both equal and our algorithm correctly determines the optimal three finger grasp for P. $\square$

5.2 Some Related Open Questions

There are several open questions related to the problem of finding optimal planar grasps. We briefly discuss these problems.

(1) Consider a variation on the above problem: Suppose we are given a simple polygon P with certain

subset of ∂P designated as "forbidden" and its complement, "feasible." Assume that the feasible parts of the polygon consists of at most K segments (the edge segment ab being allowed to be a point a ($a = b$), in the degenerate case). We are asked to find an optimal three-finger grasp of the polygon with none of the fingers on a forbidden region. Using a small variation of the above algorithm, we can solve this problem in $O(k^2 \log k)$ time—only modify the line arrangement to consist of the following triple of lines per feasible edge segment $ab \subset e$, where e is an edge of P: (1) the line containing e, (2) the line normal to e and containing a, and (3) the line normal to e and containing b. If the edge segment is a point $a \in e$ then the above situation degenerates to two lines, one containing e and the other normal e at a.

(2) We do not know whether there is a better solution for the above problem with improved complexity. For instance, it is not even clear whether there are $O(n)$ time algorithms for objects with simpler geometry, e.g., convex objects. We have an $O(n)$ solution only for what we shall refer to as *circular convex* polygonal objects. A convex polygon P will be called *circular* if there is a point c in its interior (its *center*) so that the line segment from c to a line containing an edge e and normal to e is entirely within P. For instance, the convex hull of a set of points on a circle defines a circular polygon (thus, the name). Note that in this case,

$$\bigcap_{e \in E} \mathrm{slab}(e) \neq \emptyset,$$

where E is the set of edges of P. Clearly, we can find a small neighborhood U of the center c such that active(U) is all of Q. In this case, the problem reduces to simply finding three points q_i, q_j and $q_k \in Q$ such that the residual radius of the resulting triangle is as large as possible.

We need to extend the notion of residual radius as follows: The residual radius of a triangle Δ is the *signed* radius of the largest disk centered at the origin that is either fully outside or fully inside Δ, the sign being positive or negative depending on whether the disk is inside or outside Δ respectively.

Assume that the points of Q are ordered in the anti-clockwise order as

$$q_1 > q_2 > \cdots > q_n.$$

for any point $q \in Q$ its successor, succ(q), is the point immediately following it in the clockwise order.

We start with three arbitrarily chosen distinct points, say $u_0 = q_1$, $u_1 = q_2$ and $u_2 = q_3$, for instance. At any instance, assume that, we have three points u_0, u_1 and u_2, at least two of which are distinct, and

$$u_0 \geq u_1 \geq u_2 \geq u_0$$

There are two cases to consider: (1) they are not all distinct, i.e., $u_i = u_{i+1}$ and (2) they are all distinct and the residual disk touches the edge $u_{i-1}u_i$.

In the first case, we advance the "forward point" u_i (i.e, replace u_i by succ(u_i)) with the hope of making the points distinct (this may not succeed and lead to further advancements of this kind). In the second case, we advance the "backward point u_i (i.e, replace u_i by succ(u_i)) with the hope of releasing the "limiting edge" $u_{i-1}u_i$ and thus possibly (but not always) increasing the residual radius.

The algorithm keeps advancing the forward or the backward point (as the case may be) while recording the maximal residual radius seen so far until u_0 returns to its initial position, at which point it halts and outputs the edge triple corresponding to the maximal residual radius. Since the polygon is a circular convex polygon, one can easily determine the contact points by taking the normals from the center of the polygon to each edge of the edge triple. The correctness and the complexity analysis of the algorithm can be shown in a manner similar to the discussions in section 5 of the paper by Kirkpatrick et. al. [14] and is omitted here. Note, however, that the above technique fails for arbitrary convex polygons if we relax the condition of circularity.

Note that the above technique can be easily adapted to the following problem: Given a simple polygon P and a center $c \in \mathbb{R}^2$, find a 3-finger optimal grasp of P such that the inner normals at the contact points go through c. This problem is solved by simply running the above algorithm starting with an active set, active(U) of a small open neighborhood of c. The resulting algorithm takes $O(n)$ time.

(3) Sometimes, we wish to determine not just one optimal three finger grasp but all of them. Then we may use any one of this class of optimal grasps, depending on the task at hand. Clearly, the brute force $O(n^3)$ time algorithm will succeed to do so. Note that the algorithm of the previous section cannot be easily modified into a two pass algorithm, since addition of a new point (in the process of going from one cell to an adjacent cell) may create an $O(n)$ edge triplets of residual radius ρ^*. Here, we describe an $O(n^2 \log n)$ algorithm for the special case when the object is convex.

Let P be a convex n-gon and let the possible residual radii (as in the preceding subsection) be given as

$$0 \leq \rho_1 \leq \rho_2 \leq \cdots \leq \rho_i \leq \cdots < 1.$$

We shall find the optimal residual radius ρ^* by performing a binary search on the sequence of possible residual radii. For a given value of ρ_i, we can enumerate all the edge triples that lead to a residual radius of ρ_i in $O(n^2)$ as follows: Corresponding to the possible radius value ρ_i, there are at most $O(n)$ edge pairs $(e_i,\, e_j)$'s such that the corresponding points q_i and $q_j \in Q$ on the unit circle satisfy the property that the line determined by $q_i q_j$ is tangent to a circle $C(\rho_i)$ centered at the origin and of radius ρ_i. Now for each such edge pair, we need to check in $O(n)$ time if there is another edge e_k such that $q_k \in Q \setminus \{q_i,\, q_j\}$ is mutually visible (with respect to $C(\rho_i)$) to both q_i and q_j and that

$$\mathrm{slab}(e_i) \cap \mathrm{slab}(e_j) \cap \mathrm{slab}(e_k) \neq \emptyset.$$

We can thus enumerate all the e_k's that succeed this test. The binary search only considers $O(\log n)$ different values of ρ_i's and terminates with success with the largest possible value ρ^* and enumerating all edge triples corresponding to ρ^*. It is then trivial to describe all possible three finger optimal planar grasps. Thus the algorithm has a time complexity of $O(n^2 \log n)$.

However, the algorithm applied to a nonconvex polygon leads to an $O(n^3)$-time algorithm, as in a pathological case, there may be $O(n^2)$ edge pairs to be considered for a given value of ρ_i. It is noteworthy that this algorithm is rather simple to implement and may perform well in practice. For instance, if one performs binary search on the real interval $[0, 1]$ (instead of the possible radii values), then for a random polygon this algorithm can compute in $O(n \log n \log(1/\epsilon))$ all three finger grasps whose corresponding residual radii lie in the range $[\rho,\, \rho^*]$ of size $< \epsilon$, for sufficiently small positive ϵ.

(4) We still do not know how to find optimal m-finger planar grasp ($m \geq 4$) in time better than what can be obtained by the brute force algorithm taking time $O(n^{O(m)})$. For instance, it is not even clear if there is an algorithm to compute such an optimal grasp in time $O(n^{m-1}\text{polylog } n)$. The complication arises by virtue of the torque components that one has to consider in the case when $m \geq 4$.

Some progress has been made, by modifying the problem to that of choosing an optimal set of m-finger contact points out of a preselected $O(n)$ points on the boundary of the polygon, ∂P. For instance, we have an $O(n^3 \log n)$ time algorithm to find such an optimal four finger grasp in this case. The technique employed for this case is a generalization of the preceding algorithm involving binary search. We suspect that the algorithm generalizes to m fingers ($m > 4$) and has a time complexity of $O(n^{m-1} \log n)$.

(5) In case of parallel jaw grippers and three-jaw grippers grasping an n-gon, one can compute the optimal grasps in time $O(n)$. The algorithms in these cases involve simply going around the object and trying all possible grasps [9]. It is not clear, if these grippers are comparable to multi-fingered hands in terms of how well they optimize various grasp metrics.

(6) Another problem of interest is to study the similar optimality problem in the case of "fixturing," where a polygonal object has to be fixtured by a set of toe-clamps that can be placed only at places designated by a set of toe-slots (which are usually arranged on a regular square grid) [3, 35, 19, 20]. The added difficulty arises because of the geometric constraints imposed by the toe-slots. It is easily seen that for a rectilinear object the optimal fixel (fixture element) placement can be determined in $O(n)$ time. However, the problem

seems quite difficult even when we consider a convex polygonal object.

(7) Recently, we have been able to design "reactive hands" for grasping. These algorithms operate by determining a sensor-dependent binary vector and then actuating a small set of actuators by a simple table-lookup procedure[31, 32]. It remains an intriguing open question whether it is possible to design general multi-fingered reactive hands that always find an optimal grasp.

6 Computational Issues

The study of grasp metrics suggest two kinds of algorithmic (and the related complexity) problems:

1. Computing the quality of a given grasp under the chosen grasp metric.

2. Computing the optimal grasp of an object by an m-fingered hand under the chosen grasp metric.

Henceforth, we shall refer to these two problems as 1) *Computation Problem* and 2) *Optimization Problem*, respectively. In general, the computation problem seems to have received far more attention than the optimization problem.

6.1 Computation Problem

Given: A grasp $\mathcal{G}$ by means of its associated system of wrenches: $\{\mathbf{w}_1, \ldots, \mathbf{w}_k, \mathbf{w}_{k+1}, \ldots, \mathbf{w}_n\}$, where $\mathbf{w}_1$, ..., $\mathbf{w}_k$ are bisense and the remaining $\mathbf{w}_{k+1}$, ..., $\mathbf{w}_n$ are unisense. Also given is a grasp metric, described by the finger force constraint χ, where χ is assumed to be convex, closed, compact and faithful.

Compute: The quality of the given grasp $r_\chi(\mathcal{G})$. Recall that r_χ is the radius of the largest sphere centered at the origin and contained in the feasible wrench set G_χ.

$$G_\chi = \left\{\mathbf{w} = \sum f_i \mathbf{w}_i : \chi(f_1, \ldots, f_n) = 1\right\}.$$

Thus the given grasp has a quality of $r_\chi(\mathcal{G})$ if and only if

$$\Big(\forall r \le r_\chi(\mathcal{G})\Big) \; \Big(\forall \mathbf{w} = (w_1, \ldots, w_6) \in \mathbb{R}^6\Big)$$
$$\Big[|\mathbf{w}| = r \;\; \Rightarrow \;\; \mathbf{w} \in G_\chi\Big],$$

or equivalently,

$$\Big(\forall r \le r_\chi(\mathcal{G})\Big) \; \Big(\forall \mathbf{w} \in \mathbb{R}^6\Big) \; \Big(\exists f_1, \ldots, f_n\Big)$$
$$\Big[(|\mathbf{w}| = r) \;\; \Rightarrow \;\; \mathbf{w} = \sum f_i \mathbf{w}_i$$
$$\wedge \, (\chi(f_1, \ldots, f_n) = 1\Big].$$

Thus we need to maximize $r_\chi(\mathcal{G})$ subject to the conditions described above to get the quality of the grasp.

Thus, if we assume that $\chi(f_1, \ldots, f_n)$ is given by a set of algebraic equations and inequalities of some bounded degree, then the above problem can be solved by using efficient algorithms in Tarski's elementary geometry in exponential time:

$$O\left(2^{O(n \log n)}\right),$$

in the cardinality of the wrench system n. (See Renegar [27].)

(1) Note that some simple improvements can be obtained in the special cases as follows: For instance, consider the finger force constraint defined by χ_{con}, in this case $G_{\chi_{con}}$ can be expressed as the intersection of upto $\binom{n}{6} = O(n^6)$ half spaces which define the convex hull of

$$\{\mathbf{w}_i, -\mathbf{w}_i : i = 1, \ldots, k\} \cup \{\mathbf{w}_i : i = k+1, \ldots, n\}.$$

Now considering the set of all normals from the origin to the boundary hyperplanes of these half spaces, one can compute $r_{\chi_{con}}$ by simple brute force. Note that computing these half spaces is quite trivial. For every six vectors, consider the hyperplane corresponding to their affine hull. If all the remaining points lie on one side (say positive) of the hyperplane, then we include the associated half space in our collection. The resulting algorithm takes $O(n^7)$ time and is thus polynomial. The exponent can be slightly improved by introducing additional data structure or randomization, but since we suspect that the improvement is not substantial, we shall not discuss this any further.

A similar consideration for χ_{max}, however, gives a $O(2^{O(n)})$-time algorithm, but we do not know if for the special cases where χ is given by set of linear equations and inequalities, the problem has a better complexity than the one with trivial $O(2^{O(n \log n)})$ bound. Note that since n is usually small (i.e., between 4 and 12), this complexity may be deemed acceptable.

(2) Another approach to this problem would be to probe G_χ in m distinct directions, each direction being given by a ray R going through the origin. Also assume that there is an oracle, which given R, returns $R \cap \partial G_\chi$. Thus after m probes, we can choose as an approximation for r_χ, a function of the probe values returned by the oracle (for instance, the minimum of the magnitudes). Here, we have tacitly assumed that we have no knowledge of G_χ other than what is provided by the oracle. Standard argument (similar to the one in [1]) shows that the number of probes will be exponentially large in d in order to obtain a good approximation, in general. Here, of course, $d = 6$, the dimension of G_χ.

The argument is as follows: We operate in dimension d. When the oracle is given a d-dimensional ray R, the oracle returns a unit vector $\mathbf{v}_R$, $|\mathbf{v}_R| = 1$, in the direction of R. We perform m such probes, $m \geq 13^d d^{\frac{d+3}{2}}$. Choose an ϵ

$$\epsilon \ll \frac{1}{68}\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}}.$$

If we compute an approximate residual radius of some value less than $1 - 2\epsilon$ then the oracle reveals G_χ to be a unit d-ball. If we compute an approximate residual radius of some value larger than

$$1 + 2\epsilon - \frac{1}{17}\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}}$$

then the oracle reveals G_χ to be the convex hull of the points it had produced, which has a residual radius

$$\leq 1 - \frac{1}{17}\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}},$$

thus contradicting again. By the choice of our ϵ value, our approximation must fall into either category. Thus the ϵ-approximation, for a given ϵ requires m probes such that

$$m \geq \frac{2d^2}{(c\epsilon)^{d-1/2}}.$$

Thus improving the approximation by a single bit will require increasing the number of probes by a factor of $2^{d-1/2}$.

(3) Before we leave this discussion, we note that as $r_{\tilde{\chi}_2}$ and r_{null} are given by simple linear algebraic formulations, they can be computed rather easily by matrix computation and linear programming techniques, respectively. However, $r_{\tilde{\chi}_2}$ and r_{null} are not of much help in computation (or, even in providing an approximation) involving some general χ.

6.2 Optimization Problem

Given: An object B and a robot hand with exactly m fingers. The contact type of each finger is given so that given a point $\mathbf{p} \in \partial B$, the corresponding system of wrenches

$$\Gamma(\mathbf{p}) = \{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$$

($\ell \leq 6$) can be computed easily. Also given is a grasp metric, described by the finger force constraint χ, where χ is assumed to be convex, closed, compact and faithful.

Compute: A set of m contact points $\mathbf{p}_1, \ldots, \mathbf{p}_m \in \partial B$ such that the resulting grasp $\mathcal{G}$ involving the wrench system

$$\bigcup_{i=1}^{m} \Gamma(\mathbf{p}_i)$$

has the optimal grasp strength $r^* = r^*_\chi(B) = r_\chi(\mathcal{G})$. Thus we wish to

- maximize r^*
- subject to the following condition

$$\Big(\exists \mathbf{p}_1, \ldots, \mathbf{p}_m\Big)\ \Big(\forall r \leq r^*\Big)\ \Big(\forall \mathbf{w} \in \mathbb{R}^6\Big)$$
$$\Big[(\mathbf{p}_1, \ldots, \mathbf{p}_m \in \partial B\ \wedge\ |\mathbf{w}| = r)\ \Rightarrow$$
$$\mathbf{w} \in G_\chi(\bigcup_{i=1}^{m} \Gamma(\mathbf{p}_i))\Big].$$

Thus, if we assume that $\chi(f_1, \ldots, f_n)$ is given by a set of algebraic equations and inequalities of some bounded degree and ∂B is given by piecewise algebraic surfaces, each of bounded degree and of cardinality n then the above problem can be solved by using efficient algorithms in Tarski's elementary geometry in time $O(n^{O(m)})$.

Note that if we simply try to place the fingers on each of $\binom{n}{k}$ $(1 \leq k \leq m)$ portions of the surface ∂B and exhaustively try each possibility then the complexity of the algorithm can be improved to

$$O\left(n^m 2^{O(m \log m)}\right).$$

However, it is unclear whether any further improvement for this problem can be obtained in a very general setting.

(1) When m is substantially large, however one can obtain a good approximate algorithm for the case of χ_{con}. In this case, we first choose a large number of candidate points on the surface of the object ∂B such that the points are placed fairly closely and choose as our set X the image of these points under the wrench map Γ. Thus conv $X = \tilde{G}_{\chi_{con}}$ is a good approximation of the feasible wrench set $G_{\chi_{con}}$ in the sense that the residual radius of $\tilde{G}_{\chi_{con}}$ is a close approximation of that of $G_{\chi_{con}}$:

$$r(\tilde{G}_{\chi_{con}}) \geq r(G_{\chi_{con}})(1 - \epsilon_n),$$

where ϵ_n depends on the number of candidate points n and ∂B. The rest follows from the discussion below. We start with the following algorithmic problem:

> Given a set X of n points in d-dimensional Euclidean space, whose residual radius $r(X)$ is positive, find a subset $Y \subseteq X$ of at most m points such that the following inequality holds:
>
> $$\frac{r(Y)}{r(X)} \geq \tilde{r}_d(m) = 1 - 3d\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}}.$$
>
> Here m and n are assumed to be sufficiently large, i.e. $n \geq m \geq 13^d d^{\frac{d+3}{2}}$.

We see that this problem can be solved by essentially following the ideas outlined in the proof of KMY bounds: We first choose a set Y' of at most m/d points on the surface of the unit ball such that the residual radius of Y' is no smaller than $\tilde{r}_d(m)$. We can then determine a set $Y \subseteq X$ of at most m points such that for some $\lambda_{\min} \geq r(X)$, the convex hull of Y contains the set of points

$$\lambda_{\min} Y' = \{\lambda_{\min}\, q : q \in Y'\}.$$

Thus

$$r(Y) \geq r(\lambda_{\min} Y') \geq \lambda_{\min} \tilde{r}_d(m) \geq r(X)\tilde{r}_d(m).$$

The points of Y' are chosen as follows: Let C be the d-dimensional cube comprising the points $(y_1, \ldots, y_d)$ with $|y_i| \leq 1$ for $i = 1, \ldots, d$. On each face of C, we place a $k \times k \times \cdots \times k$ ($(d-1)$ times) grid, with k taking the value

$$\left\lfloor \left(\frac{m}{2d^2}\right)^{\frac{1}{(d-1)}} \right\rfloor.$$

Let

$$Y' = \left\{\vec{op} \cap \mathcal{S}^{d-1} : p \text{ is a grid point}\right\}.$$

Thus $|Y'| \leq 2dk^{d-1} \leq m/d$. For each $q \in Y'$, we determine an appropriate set $X_q \subseteq X$ of at most d points such that

$$\vec{oq} \cap \partial\,(\text{ConvexHull}(X)) \in \text{ConvexHull}(X_q);$$

thus for some λ_q,

$$\lambda_q q \in \text{ConvexHull}(X_q).$$

Let Y be

$$Y = \bigcup_{q \in Y'} X_q,$$

with $\lambda_{\min}$ taking the value $\min_{q \in Y'} \lambda_q$. Evidently, $\lambda_{\min} \geq r(X)$.

Note that $|Y| \leq m$, and

$$\lambda_{\min} Y' \subseteq \text{ConvexHull}(Y).$$

This demonstrates the correctness of the algorithm, since we know that the residual radius of Y' is bounded from below by $\tilde{r}_d(m)$. (See the proof for the KMY bounds).

In order to complete the algorithm, we show how to efficiently compute the set X_q (for any point q) using the following linear programming formulation. Let $X = \{p_1, p_2, \ldots, p_n\}$. Without loss of generality, we assume that the points of X are in *general position*, i.e., *at most d points of X may lie on any $(d-1)$ dimensional hyperplane*. If not, the original points of X may be perturbed using generic perturbation methods (see, for example, [34]); the following discussions still apply *mutatis mutandis*. Define the $d \times n$ matrix $\mathbf{A}$ whose j^{th} column consists of the coordinates of the point p_j. Corresponding to the point q, define a column d-vector $\mathbf{b}$. The linear programming problem (LP) is given as follows:

- GIVEN: A $d \times n$ matrix $\mathbf{A}$ and a column d-vector $\mathbf{b}$.
- SOLVE:

$$\begin{array}{lrcl} \text{minimize} & -\lambda & & \\ \text{subject to} & \mathbf{A}\mathbf{x} & = & \lambda\mathbf{b} \\ & \mathbf{e}^{\mathrm{T}}\mathbf{x} & = & 1 \\ & \mathbf{x} & \geq & \mathbf{0} \\ & \lambda & \geq & 0, \end{array}$$

where $\mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{T}}$, $\mathbf{e} = (1, \ldots, 1)^{\mathrm{T}}$ and $\mathbf{0} = (0, \ldots, 0)^{\mathrm{T}}$ are column n-vectors.

Let $\mathbf{x}^*$, λ^* be an optimal solution of (LP). Then $\lambda^* > 0$ is the maximum value of λ such that

$$\lambda^* q = \sum_{i=1}^{n} x_i^* p_i,$$

with $\sum_{i=1}^{n} x_i^* = 1$, and $x_i^* \geq 0$.

Now consider the following *dual* of the (LP), which will be referred to as (DLP):

$$\begin{array}{lrcl} \text{maximize} & y_{d+1} & & \\ \text{subject to} & a_{1,1}y_1 + \cdots + a_{d,1}y_d + y_{d+1} & \leq & 0 \\ & a_{1,2}y_1 + \cdots + a_{d,2}y_d + y_{d+1} & \leq & 0 \\ & & \vdots & \vdots \\ & a_{1,n}y_1 + \cdots + a_{d,n}y_d + y_{d+1} & \leq & 0 \\ & -b_1y_1 - \cdots - b_dy_d & \leq & -1 \end{array}$$

This problem can be solved in $O(3^{d^2}n)$ time by using Clarkson-Dyer's improvement on Megiddo's multidimensional search technique [7, 8, 17]. Let us now see how to recover the solution to the original problem.

Clearly both (LP) and (DLP) have optimal solutions. Let an optimal solution for (DLP) be

$$\mathbf{y}^* = (y_1^*, \ldots, y_d^*, y_{d+1}^*).$$

Let $I_q \subseteq \{1..n\}$ be the set of all the indices j such that

$$\mathbf{a}_j \cdot \mathbf{y}^* = a_{1,j}y_1^* + \cdots + a_{d,j}y_d^* + y_{d+1}^* = 0.$$

where $\mathbf{a}_j = (a_{1,j}, \cdots, a_{d,j}, 1)^{\mathrm{T}}$. By the Complementary Slackness Theorem (see [6]), this implies that for all $i = 1, \ldots, n$, if $x_i^* > 0$ then $i \in I_q$. By virtue of our non-degeneracy hypothesis about the points of X, we see that $|I_q| \leq d$. We now claim that $X_q = \{p_j : j \in I_q\}$ can serve as a desired solution. Clearly, $X_q \subseteq X$, has at most d points and

$$\vec{oq} \cap \partial(\text{ConvexHull}(X)) \in \text{ConvexHull}(X_q)).$$

Note that even if the original set had been perturbed (by a sufficiently small amount) the set X_q chosen from the unperturbed set X still provides the desired solution.

To summarize:

Theorem 7 *For $n \geq m \geq 13^d d^{(d+3)/2}$, we can find an m-fingered grasp $\mathcal{G}$ of an object B such that*

$$\frac{r_{\chi_{con}}(\mathcal{G})}{r^*} \geq (1-\epsilon_n)\left(1 - 3d\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}}\right),$$

in time $O(3^{d^2}mn)$. Here n is the number of candidate points and ϵ_n is a small number that depends on n and ∂B. □

7 Bibliographic Notes

The general framework describing the connection between grasping and convexity theory is due to Mishra, Schwartz and Sharir [21]. The results relating form and force/torque closure appear in the paper by Mishra and Silver [22] and the results relating immobility and force/torque closure appear in the paper by Mishra and Teichmann [23]. Some of the other related results are taken from our earlier work [18, 23].

The grasp metric of section 2 is partly based on our earlier work [14]. For the sake of concreteness, only the finger force constraint χ_{con} was discussed. Ferrari and Canny [9] subsequently suggested that in some situations χ_{max} condition may be of more practical interest. Independently, Li and Sastry [15] proposed a metric that corresponds to the conditions referred to here as $\tilde{\chi}_2$ and χ_2. Trinkle [33] studied the problem of computing with the grasp metric, r_{null}, for small number of fingers. The relations between r_{null} and $r_{\chi_{con}}$ described here has not appeared elsewhere. The grasp metric based on the nasty finger model and the result proving the weakness of a seven finger grasp is due to Walter Meyer and was motivated by our original (KMY) grasp metric.

The results involving Q.S.T. are due to Bárány, Katchalski and Pach [2] and the generalizations motivated by the study of grasp metrics are due to Kirkpatrick, Mishra and Yap [14]. The approximate algorithm in the general case and appearing in the last section is based on the results of Kirkpatrick, Mishra and Yap [14]. The results related to optimal three finger planar grasp are jointly with Marek Teichmann of N.Y.U.

An algorithmic study of the modular fixturing problem in manufacturing was initiated by the present author [19, 20]. Recently, some important further progress in this area, specially with an emphasis on the quality of the fixturing, has been made by Brost, Goldberg, Wong and Zhuang [3, 35]. The study of reactive robotics algorithm in general as well as in the context of grasping, has been initiated by Teichmann and Mishra [31, 32].

Appendix: Geometric Terminology

A.1 Linear Spaces and Convexity

A d-dimensional space, $\mathbb{R}^d$, equipped with the standard linear operations, is said to be a *linear space*.

1. A *linear combination* of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ from $\mathbb{R}^d$ is a vector of the form
$$\lambda_1\mathbf{x}_1 + \cdots + \lambda_n\mathbf{x}_n,$$
where $\lambda_1, \ldots, \lambda_n$ are in $\mathbb{R}$.

2. An *affine combination* of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ from $\mathbb{R}^d$ is a vector of the form
$$\lambda_1\mathbf{x}_1 + \cdots + \lambda_n\mathbf{x}_n,$$
where $\lambda_1, \ldots, \lambda_n$ are in $\mathbb{R}$, with $\lambda_1 + \cdots + \lambda_n = 1$.

3. A *positive (linear) combination* of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ from $\mathbb{R}^d$ is a vector of the form
$$\lambda_1\mathbf{x}_1 + \cdots + \lambda_n\mathbf{x}_n,$$
where $\lambda_1, \ldots, \lambda_n$ are in $\mathbb{R}_{\geq 0}$.

4. A *convex combination* of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ from $\mathbb{R}^d$ is a vector of the form
$$\lambda_1\mathbf{x}_1 + \cdots + \lambda_n\mathbf{x}_n,$$
where $\lambda_1, \ldots, \lambda_n$ are in $\mathbb{R}_{\geq 0}$ with $\lambda_1 + \cdots + \lambda_n = 1$.

By convention, we allow the empty linear combination (with $n = 0$) to take the value $\mathbf{0}$. We also assume that the empty linear combination is neither an affine combination nor a convex combination.

Note that affine, positive and convex combinations are all linear combinations, and a convex combination is both affine and positive combinations.

A nonempty subset $L \subseteq \mathbb{R}^d$ is said to be a

1. *linear subspace*: if it is closed under linear combinations;
2. *affine subspace (or, flat)*: if it is closed under affine combinations;
3. *positive set (or, cone)*: if it is closed under positive combinations; and
4. *convex set*: if it is closed under convex combinations.

The intersection of any family of linear subspaces of $\mathbb{R}^d$ is again a linear subspace of $\mathbb{R}^d$. For any subset M of $\mathbb{R}^d$, the intersection of all linear subspaces containing M (i.e. the smallest linear subspace containing M) is called the *linear hull* of M (or, the linear subspace *spanned* by M), and is denoted by lin M.

Similarly, the intersection of any family of affine subspaces, or positive sets or convex sets of $\mathbb{R}^d$ is again, respectively, an affine subspace or positive set or convex set. Thus for any subset M of $\mathbb{R}^d$, we can define

1. the *affine hull* (denoted by aff M) to be the smallest affine subspace containing M,
2. the *positive hull* (denoted by pos M) to be the smallest positive set containing M, and
3. the *convex hull* (denoted by conv M) to be the smallest convex set containing M.

They are also called, respectively, the affine subspace, positive set and convex set *spanned* by M.

Equivalently, the linear hull lin M can be defined to be the set of all linear combinations of vectors from M. Similarly, the affine hull aff M (respectively, the positive hull pos M, the convex hull conv M) can be defined to be the set of all affine (respectively, positive, convex) combinations of vectors from M.

A set $\mathbf{x}_1, \ldots, \mathbf{x}_n$ of n vectors from $\mathbb{R}^d$ is said to be *linearly independent* if a linear combination

$$\lambda_1 \mathbf{x}_1 + \cdots + \lambda_n \mathbf{x}_n$$

can only have the value $\mathbf{0}$, when $\lambda_1 = \cdots = \lambda_n = 0$; otherwise, the set is said to be *linearly dependent*.

A set $\mathbf{x}_1, \ldots, \mathbf{x}_n$ of n vectors from $\mathbb{R}^d$ is said to be *affinely independent* if a linear combination

$$\lambda_1 \mathbf{x}_1 + \cdots + \lambda_n \mathbf{x}_n \quad \text{with } \lambda_1 + \cdots + \lambda_n = 0$$

can only have the value $\mathbf{0}$, when $\lambda_1 = \cdots = \lambda_n = 0$; otherwise, the set is said to be *affinely dependent*.

A *linear basis* of a linear subspace L of $\mathbb{R}^d$ is a set M of linearly independent vectors from L such that $L = \text{lin } M$. The dimension dim L of a linear subspace L is the cardinality of any of its linear basis.

An *affine basis* of an affine subspace A of $\mathbb{R}^d$ is a set M of affinely independent vectors from L such that $A = \text{aff } M$. The dimension $\dim A$ of an affine subspace A is one less than the cardinality of any of its affine basis.

Let C be any convex set. Then by *d-interior* of C, denoted $\text{int}_d\, C$, we mean the set of points p such that, for some d-dimensional affine subspace, A, p is interior to $C \cap A$ relative to A. If c is the dim aff C, then by an abuse of notation, we write int C to mean $\text{int}_c\, C$.

For subsets A and B of $\mathbb{R}^d$ and λ real define the *(Minkowski) sum* of A and B by

$$A + B = \{a + b : a \in A, b \in B\},$$

and let λA be

$$\lambda A = \{\lambda a : a \in A\}.$$

We shall write $A \oplus B$ instead of $A + B$ if A and B are contained in subspaces of $\mathbb{R}^d$ for which the usual direct sum exists: $A \oplus B$ is then called the *direct sum* of A and B. Call C *directly irreducible* if there is no representation of C of the form $A \oplus B$ where both A and B are different from the origin. By a decomposition theorem of Gruber, we have the result that each convex body C can be represented in the form $C_1 \oplus \cdots \oplus C_m$ where $C_1, \ldots, C_m$ are directly irreducible. Such a representation is unique modulo the order of the summands.

A.2 Screw Theory

A *screw* is defined by a straight line in three-dimensional Euclidean space, called, its *screw-axis* and an associated *pitch*, p. A screw is represented by a six-dimensional vector, $\mathrm{s} = (S_1, S_2, S_3, S_4, S_5, S_6)$, known as the *screw coordinates*. The screw coordinates are interpreted in terms of the *Plücker line coordinates*, (L, M, N, P, Q, R), of the screw axis, as follows:

$$\begin{aligned} L &= S_1, \\ M &= S_2, \\ N &= S_3, \\ P &= S_4 - pS_1, \\ Q &= S_5 - pS_2, \\ R &= S_6 - pS_3, \end{aligned}$$

where L, M and N are proportional to the direction cosines of the screw axis, and P, Q and R are proportional to the moment of the screw axis about the origin of the reference frame (i.e. the cross product of a vector from the origin to a point on the axis and a unit vector, directed along the screw axis). The pitch of the screw is then given by

$$p = \frac{S_1 S_4 + S_2 S_5 + S_3 S_6}{S_1^2 + S_2^2 + S_3^2},$$

and the magnitude of the screw is given by

$$|\mathrm{s}| = \begin{cases} \sqrt{S_1^2 + S_2^2 + S_3^2}, & \text{if } p < \infty; \\ \sqrt{S_4^2 + S_5^2 + S_6^2}, & \text{if } p = \infty. \end{cases}$$

A *unit screw* is a screw with unit magnitude. Scalar multiplication and vector addition are valid for infinitesimal screws, and the screws are closed under these operations. Thus the six-dimensional space of infinitesimal screws forms a vector space.

Sometimes, we simply consider the 2-norms of a screw (as a six-dimensional vector), disregarding its pitch:

$$|\mathrm{s}|_2 = \sqrt{\sum_{i=1}^{6} S_i^2}.$$

Given two screws $\mathrm{s}' = (S_1', S_2', S_3', S_4', S_5', s_6')$ and $\mathrm{s}'' = (S_1'', S_2'', S_3'', S_4'', S_5'', S_6'')$, we define their *virtual coefficient* as

$$\mathrm{s}' \odot \mathrm{s}'' = S_1' S_4'' + S_2' S_5'' + S_3' S_6'' + S_4' S_1'' + S_5' S_2'' + S_6' S_3''.$$

Note that the operation '$\odot$' is a commutative operation from $\mathbb{R}^6 \times \mathbb{R}^6$ into $\mathbb{R}$.

Two screws s' and s'' are said to be

1. *reciprocal*: if their virtual coefficient is zero, i.e. $s' \odot s'' = 0$,
2. *repelling*: if their virtual coefficient is strictly positive, i.e. $s' \odot s'' > 0$, and
3. *contrary*: if their virtual coefficient is strictly negative, i.e. $s' \odot s'' < 0$.

An ensemble of screws is known as a *screw system*, and is defined by a set of $n \leq 6$ independent *basis screws*. The *order* of a screw system is equal to the number of basis screws required to define it; such a system is also called an *n-system*. The order of a screw system reciprocal to an n-system is $(6 - n)$.

With an infinitesimal rigid motion of an object in three-dimensional Euclidean space there is an associated screw called *twist* such that the body rotates about and translates along its screw axis. The screw coordinates of a twist are given by $\mathbf{t} = (T_1, T_2, T_3, T_4, T_5, T_6)$, where the first three components T_1, T_2 and T_3 correspond to the angular displacement (or angular velocity), $\overline{\omega}$, of the body and the last three components T_4, T_5 and T_6 correspond to the translational displacement (or translational velocity), $\overline{v}$, of a point fixed in the body and lying at the origin of the coordinate system. The pitch of the twist is given by

$$p = \frac{\overline{\omega} \cdot \overline{v}}{\overline{\omega} \cdot \overline{\omega}}.$$

The pitch of the twist is the ratio of the magnitude of the velocity of a point on the twist axis to the magnitude of the angular velocity about the twist axis. If the pitch of a twist is zero then the twist corresponds to a pure rotation, and if the pitch of a twist is infinite then the twist corresponds to a pure translation. The magnitude of the twist is given by

$$|\mathbf{t}| = \begin{cases} \|\overline{\omega}\|_2, & \text{if } p < \infty; \\ \|\overline{v}\|_2, & \text{if } p = \infty. \end{cases}$$

Similarly, with any system of forces and torques acting on a rigid object in three-dimensional Euclidean space there is an associated screw called *wrench* such that the system of forces and torques can be replaced by an equivalent system of single force along the wrench axis and a torque about the same wrench axis. The screw coordinates of a wrench are given by $\mathbf{w} = (W_1, W_2, W_3, W_4, W_5, W_6)$, where the first three components W_1, W_2 and W_3 correspond to the resultant force, $\overline{f}$, acting on the body along the wrench axis and the last three components W_4, W_5 and W_6 correspond to the resultant torque, $\overline{\tau}$, acting on the body about the wrench axis. The pitch of the wrench is given by

$$p = \frac{\overline{f} \cdot \overline{\tau}}{\overline{f} \cdot \overline{f}}.$$

The pitch of the wrench is the ratio of magnitude of the torque acting about a point on the axis to the magnitude of the force acting along the axis. If the pitch of a wrench is zero then the wrench corresponds to a pure force, and if the pitch of a wrench is infinite then the wrench corresponds to a pure moment. The magnitude of the wrench is given by

$$|\mathbf{w}| = \begin{cases} \|\overline{f}\|_2, & \text{if } p < \infty; \\ \|\overline{\tau}\|_2, & \text{if } p = \infty. \end{cases}$$

Note that the virtual coefficient of a twist $\mathbf{t} = (\overline{\omega}, \overline{v})$ and a wrench $\mathbf{w} = (\overline{f}, \overline{\tau})$ is

$$\mathbf{w} \odot \mathbf{t} = \overline{f} \cdot \overline{v} + \overline{\tau} \cdot \overline{\omega},$$

the rate of change of work done by the wrench $\mathbf{w}$ on a body moving with the twist $\mathbf{t}$.

If a twist $\mathbf{t}$ is reciprocal to a wrench $\mathbf{w}$, then the wrench does no work when the body is displaced infinitesimally by the twist. Thus for two reciprocal screws, a twist about one of the screws is possible while the body is being constrained about the other screw. Similarly, if $\mathbf{t}$ is repelling to $\mathbf{w}$, then positive work is done by the constraining wrench when the body is displaced infinitesimally by the twist. This implies that the twist can be accomplished, but then the contact of the wrench will be definitely broken. Lastly, if $\mathbf{t}$ is contrary to $\mathbf{w}$, then negative (virtual) work must be done by the constraining wrench when the body is displaced infinitesimally by the twist. This implies that such a displacement is impossible, if we assume that the objects being considered are all rigid.

For a given wrench system acting on a body, we say that the body has *total freedom*, if the body can undergo all possible twists, without breaking the contacts associated with the wrenches; we also say that the body has *total constraint*, if the body cannot undergo any twist, without breaking the contacts; otherwise, we say that the body has *partial constraint*.

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