Quantitative Steinitz's Theorems with Applications to Multifingered Grasping[1]

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Abstract

We prove the following quantitative form of a classical theorem of Steinitz: Let m be sufficiently large. If the convex hull of a subset S of Euclidean d-space contains a unit ball centered on the origin then there is a subset of S with at most m points whose convex hull contains a solid ball also centered on the origin and having residual radius

$$1 - 3d\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}}.$$

The case $m = 2d$ was first considered by Bárány, Katchalski and Pach (1982). We also show an upper bound on the achievable radius: the residual radius must be less than

$$1 - \frac{1}{17}\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}}.$$

These results have applications in the problem of computing the so-called closure grasps *by an m-fingered robot hand. The above quantitative form of Steinitz's theorem gives a notion of* efficiency *for closure grasps. The theorem also gives rise to some new problems in computational geometry. We present some efficient algorithms for these problems, especially in the two dimensional case.*

September 17, 1991

[1] Supported by NSF Grants #DCR-84-01898 and #DCR-84-01633, ONR Grant #N00014-89-J3042. NSF Grant Subcontract #CMU-406349-55586. Chee Yap is supported in part by the German Research Foundation (DFG), and by the ESPRIT II Basic Research Actions Program of the EC under contract No. 3075 (project ALCOM)

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1 Introduction

Carathéodory's theorem [2] states that given a subset S of Euclidean d-space E^d, any point inside its convex hull is also inside the convex hull of some subset of S with at most $d+1$ points. Steinitz's theorem [13,14,15] states that given a subset $S \subseteq \mathrm{E}^d$, any point in the interior of its convex hull is also in the interior of the convex hull of some subset of S with at most $2d$ points. Bárány, Katchalski and Pach [1] showed that the following quantitative version of Steinitz's theorem holds.

Theorem 1.1 Quantitative Steinitz's Theorem
For any positive d there is a constant $r = r(d) > d^{-2d}$ such that given any set $S \subseteq \mathrm{E}^d$ of points in d-space whose convex hull contains the unit ball centered at the origin o, there is a subset $X \subseteq S$ with at most 2d points whose convex hull contains a ball centered at o with radius $r(d)$. □

In fact, [1] notes that $r(d) > c(2ed)^{-\lfloor d/2 \rfloor} d^{-2}$ for some constant c. In this paper, we generalize this quantitative Steinitz's theorem, and study various algorithmic questions related to it.

We introduce the following terminology: For any set $S \subseteq \mathrm{E}^d$, let the *residual ball* of S refer to the largest (closed) ball $\mathrm{B}(S)$ centered at the origin o such that the interior of $\mathrm{B}(S)$ is either fully outside or fully contained inside the convex hull of S. The *residual radius* of S, denoted $r(S)$ is the *signed* radius of this residual ball, where the sign is zero if $\mathrm{B}(S)$ is a point, otherwise the sign is positive or negative depending respectively on whether $\mathrm{B}(S)$ lies inside or outside the convex hull. Let $r_d(m, S)$ [or, $r(m, S)$, if d is clear from the context] denote the largest residual radius of a subset X of S with at most m points. Let $r_d(m)$ denote the minimum value of $r_d(m, S)$, as S ranges over all subsets $S \subseteq \mathrm{E}^d$ with $r(S) \geq 1$. Hence for $m \geq 2d$, the result of Bárány, Katchalski and Pach shows that $d^{-2d} \leq r_d(m) \leq 1$. Here, we derive tighter upper and lower bounds for $r_d(m)$. Note that the notation $r(d)$ in the Quantitative Steinitz's theorem above is simply the case of $r_d(2d)$.

Application in Robotics. Our interest in these theorems comes from the study of robot hand grasps. We are interested in hands with m frictionless (point-)fingers. A *grasp* in this model consists of m points on the boundary of the body that we want to grasp. To actually grasp the body, we must then specify forces to be applied at these m (grasp) points. A desirable notion of grasping is that of a *closure grasp* (see for example [8,10,11]). Intuitively, a closure grasp has the ability to respond to any external force or torque by applying appropriate forces at the grasp points. The quantitative Steinitz theorem gives us a measure of the efficiency of such closure grasps. Roughly speaking, the *efficiency* of the grasp is given by the ratio of largest external force-torque that can be resisted by applying at most unit forces at each of the grasp points; so a ratio of one corresponds to the most efficient grasp. The quantity $r_d(m)$ in the quantitative Steinitz's theorems gives this efficiency directly.

Computational problems. These theorems naturally lead to new problems in computational geometry. For instance, given a finite set S of points, and a number m, we want to find an m-subset of S that achieves the residual radius $r(m, S)$. We will present algorithms for such problems. Here our strongest results are in two dimensions.

The rest of this paper is organized as follows. In section 2, we explain more precisely the connection between quantitative Steinitz's theorem and grasping. Sections 3 and 4 prove the generalized Steinitz's theorems. Sections 5 and 6 present algorithms for computational versions of the generalized Steinitz's theorems. We conclude in section 7.

2 Application to Multifingered Positive Grasps

We refer to [12] for a general survey of the field and to [8,9,10,11] for the theory of robot grasping as used in this work. Consider an idealized dextrous hand, consisting of $m \geq 2$ independently movable force-sensing fingers. These fingers can only contact objects at their tips, and can thus be represented as points in three-dimensional space. The goal is to grasp a (closed, bounded and connected) rigid object K. A finger can only apply a force on the object K at the point of contact with K. We assume that the points of contact are *non-singular* (i.e., the surface Σ of K has a unique surface normal at each such point) and frictionless, and hence the force can only be applied along the surface-normal at the point of contact, directed inward into the object K. An interesting task for such a hand is that of *grasp selection* for a given object K; by a *grasp* we mean a set of m points on the boundary of K. We also describe such grasps in our model as *positive* because the fingers can only push into the body, but not pull at the body – as might happen if we postulate "sticky" fingers.

The object K to be grasped is assumed to have a piece-wise smooth boundary Σ. Assume that the grasp points are to be chosen from a given subset S of the non-singular points of Σ. For example, S may consist of all non-singular points of Σ (by definition, the surface normals at non-singular points are uniquely defined). Or again, S may be a set of finitely many pre-selected points. For any point $\mathbf{r}$ in S, let $\mathbf{n}(\mathbf{r})$ denote the unit surface normal (directed inward) at $\mathbf{r}$. Define the function Γ mapping S into the six-dimensional force/torque space as follows:

$$\begin{aligned} \Gamma \quad &: \quad S \to \mathrm{R}^6 \\ &: \quad \mathbf{r} \mapsto [\mathbf{n}(\mathbf{r}), \mathbf{r} \times \mathbf{n}(\mathbf{r})] \end{aligned}$$

where $\times$ here denotes the vector cross-product of 3-dimensional vectors. Essentially, Γ maps $\mathbf{r}$ to the point $\Gamma(\mathbf{r})$ in the force/torque space that represents the effects of applying a unit force at $\mathbf{r}$ in the direction $\mathbf{n}(\mathbf{r})$.

If $X \subseteq S$ is a set of m-points, we call X an m-finger *closure grasp* if the interior of the convex hull of $\Gamma(X) = \{\Gamma(\mathbf{r}) : \mathbf{r} \in X\}$ contains the origin o. It is shown in [8] that, for some $m \leq 12$ an m-finger closure grasp of K exists if and only if

$$o \in \text{interior}(\text{convexhull}\ (\Gamma(S))).$$

Moreover, if Σ is not a surface of revolution, the above condition is always satisfied with S equal to the set of non-singular points of Σ. For polyhedral objects, Mishra, Schwartz and Sharir [8] also gave an algorithm to find a 12-finger closure grasp in linear time. However, in the absence of any measure of "goodness" for closures grasps, the synthesized grasp may not be very robust. The motivation for our work is to quantify the goodness of closure grasps and to synthesize provably good closure grasps.

One criterion for goodness is the "efficiency" of a grasp, which measures the amount of external force and torque that can be resisted by applying at most a unit of force at each grasp point. This is precisely the value $r_6(m, \Gamma(S))$. To see this, note that if we choose m points in

$\Gamma(S)$ with residual radius r then any force/torque vector $\mathbf{v}$ whose Euclidean norm is at most r can be written as a convex combination of the m chosen points. So if $\mathbf{v}$ is any external force/torque that is applied to the body K, and $\mathbf{v}$ lies in the residual ball of radius r, we can counter this external force/torque by applying suitable forces (of magnitude at most 1) at the grasp points such that these forces sum to $-\mathbf{v}$; hence, we maintain the body in equilibrium.

These concepts can be specialized to the case where K is a planar body in which case the force/torque space is 3-dimensional. The number of fingers (12) for closure grasps can be reduced to 6 in this case [8] .

3 Quantitative Steinitz's Theorem in 2 Dimensions

Let $S \subseteq \mathrm{E}^2$ be a subset of the Euclidean plane and P its convex hull. Without loss of generality, we may assume that P has at least four vertices. Also, it is assumed that a unit disc B_1 centered about the origin o is contained inside P. In general, let B_ϵ denote the closed disc of radius $\epsilon > 0$ centered at the origin. Our goal is to develop techniques, given S and m, for choosing a set X of at most m points from S so that the residual radius of X is maximized.

Lemma 3.1 *Given S as above, for any integer $k \geq 3$, we can find a subset X of at most $2k-1$ points of S such that the convex hull of X contains B_δ with $\delta = \cos(\frac{\pi}{k})$.*

PROOF.
Take k equally spaced rays from o, making sure that one of them pass through a vertex of P (the convex hull of S). Let these rays intersect the unit circle centered at o at the points v_1, $\ldots$, v_k. For each ray, if it intersects a vertex of P then we choose that vertex and if it intersects an edge, we choose the two vertices of that edge. Thus we choose at most $2k-1$ points of S, forming the subset $X \subseteq S$. Clearly, the convex hull of X contains the points v_1, $\ldots$, v_k and hence, it contains the B_δ with $\delta = \cos(\pi/k)$. $\square$

We now show that this bound is asymptotically tight. It is important to note that this bound is achieved by choosing vertices of the convex hull of S.

Theorem 3.2 *For all $m > 4$, we have*

$$\frac{3\pi^2}{2(m+1)^2} < 1 - r_2(m) < \frac{2\pi^2}{m^2}.$$

PROOF.
The upper bound on $1 - r_2(m)$ comes from the previous lemma which shows that $r_2(m) > \cos(2\pi/m)$, and from the fact that $\cos x > 1 - x^2/2$.

For the lower bound, we let S be the vertices of a regular $(m+1)$-gon that just contains the unit disc B_1. Then the omission of any point of S gives a residual disc of radius

$$\frac{\cos(\frac{2\pi}{m+1})}{\cos(\frac{\pi}{m+1})}.$$

Thus $r_2(m)$ is upper bounded by this radius,

$$\begin{aligned} r_2(m) &< \frac{1 - \frac{2\pi^2}{(m+1)^2} + \frac{2\pi^4}{3(m+1)^4}}{1 - \frac{\pi^2}{2(m+1)^2}} \\ &= 1 - \frac{3\pi^2}{2(m+1)^2 - \pi^2} + \frac{4\pi^4}{3(m+1)^2(2(m+1)^2 - \pi^2)} \\ &< 1 - \frac{3\pi^2}{2(m+1)^2}. \quad \square \end{aligned}$$

The special case where $m = 4$ is of particular interest. We now give some special arguments for this. Starting with S as above, the preceding lemma shows how to choose at most 5 points of S whose convex hull contains the disc $\mathrm{B}_{\cos(\pi/3)} = \mathrm{B}_{1/2}$. It is not hard to argue that one of these 5 points has the property that its two neighboring points span an angle of at most 144° about the origin o and hence if we delete this point, the remaining four points has a residual radius of at least $(\cos 72°)/2 > 0.15$. We can do better with the following argument.

Theorem 3.3

$$0.30 < \sin 18° \le r_2(4) \le \frac{\cos 72°}{\cos 36°} < 0.38.$$

PROOF.
This upper bound is achieved by the regular pentagon (which is the special case $m = 4$ of the proof in the previous theorem showing that for S, given by the vertices of a regular $(m+1)$-gon, $r(m, S) = \cos(\frac{2\pi}{m+1})/\cos(\frac{\pi}{m+1})$.)

As before, let P denote the convex hull of S. For the lower bound, fix β to be any angle between 0 and 60°. For any vertex v_0 of P, we define its *forbidden zone* which consists of two disjoint cones, each spanning an angle of 2β at the origin o, and such that the two bisectors of these cones together with the ray $\overrightarrow{ov_0}$ are equally spaced at 120° apart. See figure 1.

A vertex v_0 of P is *bad* if there is another vertex v_1 of P that lies in its forbidden zone. First, if v_0 is bad, then we can choose three other vertices v_1, v_2, v_3 as follows. Let v_1 be a vertex of P that lies in the forbidden zone of v_0. Let R be the ray originating from o and bisecting the larger of the two angles defined by the two rays from o through v_0 and v_1, respectively. Let v_2, v_3 be the endpoints of the edge of P that R intersects. It is not hard to see that the quadrilateral $v_0v_1v_2v_3$ contains a circle of radius at least

$$\sin\left(30° - \frac{\beta}{2}\right).$$

Now suppose that P has no bad vertices. Assume that v_0 is vertically above o and the two forbidden cones C_1, C_2 of v_0 are bounded by the rays $\overrightarrow{oR_1}, \overrightarrow{oR_2}$ and $\overrightarrow{oR_3}, \overrightarrow{oR_4}$, respectively, where the R_i's are points on the unit circle. Since v_0 is not bad, each ray in C_1 intersects a common edge of P, say, v_1v_2 and similarly, each ray in C_2 intersects a common edge of P, say, v_3v_4. See figure 2.

First suppose that the angle $\angle(v_1ov_4)$ (we always measure angles clockwise from v_1 to v_4 in this notation) is at most $(120° + \beta)$. Then it is easy to see that the residual radius of $v_1v_2v_3v_4$ is at least $\sin(30° - \frac{\beta}{2})$.

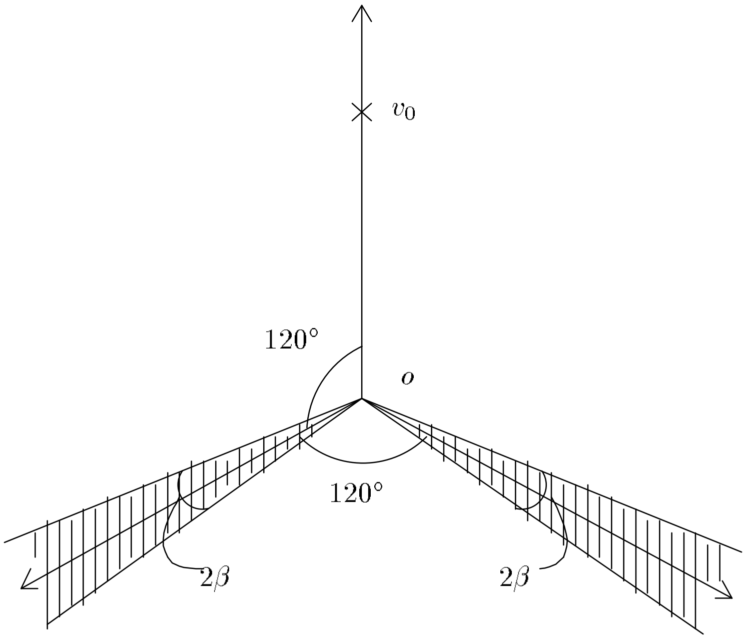

Figure 1: Forbidden zone of v_0 is shaded

Hence assume that the angle $\angle(v_1 o v_4) > (120° + \beta)$. Without loss of generality, assume that $\angle(R_1 o v_1) \leq \angle(v_4 o R_4)$. Then we have

$$\angle(R_1 o v_1) \leq (60° - \frac{3\beta}{2}), \quad \angle(R_3 o v_1) \leq (180° - \frac{3\beta}{2}).$$

Thus the distance from o to the line through v_1, R_3 is at least $\sin(\frac{3\beta}{4})$. It is easy to see that the distance from o to the line through v_0, v_1 (respectively, v_0, R_4) is at least $\sin(30° + \frac{\beta}{2})$. The distance from o to the line through R_3, R_4 is at least 1. We conclude that the residual radius of $v_0 v_1 v_3 v_4$, which is at least the residual radius of $v_0 v_1 R_3 R_4$, is at least

$$\min\left\{\sin\left(30° - \frac{\beta}{2}\right), \sin\frac{3\beta}{4}\right\},$$

where $0 \leq \beta \leq 60°$. We choose $\beta = 24°$ (i.e., $30° - \frac{\beta}{2} = \frac{3\beta}{4}$) to maximize this expression. This proves our lower bound. $\square$

4 Quantitative Steinitz's Theorems in Higher Dimensions

We now consider the d-dimensional case for $d > 2$. The techniques are slightly weaker than the 2-dimensional case.

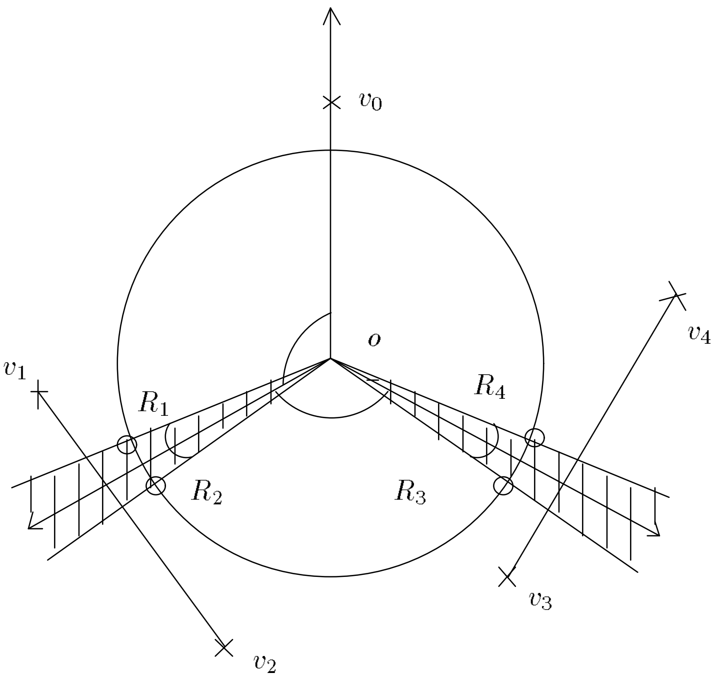

Figure 2: Case of no bad vertices.

4.1 Lower Bound

We first give a lower bound for $r_d(m)$ for sufficiently large m (in particular, for all $m \geq 13^d d^{\frac{d+3}{2}}$). Thus, m is chosen to be large enough to guarantee that

$$k = \left\lfloor \left(\frac{m}{2d^2}\right)^{1/(d-1)} \right\rfloor$$

takes integral values, greater than $\lceil 11\sqrt{d} \rceil$.

Lemma 4.1 *For any set $S \subseteq \mathrm{E}^d$ whose convex hull contains the unit ball B^d centered at the origin o, we can find a set $X \subseteq S$ of at most m points with residual radius*

$$r(X) \geq 1 - 3d\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}}, \quad \textit{for all } m \geq 13^d d^{\frac{d+3}{2}}.$$

PROOF.

Let k be defined as a function of d and m, as before. It suffices to show that

$$r(X) \geq 1 - \frac{97}{48}\frac{d}{(k-1)^2} > 1 - \frac{3d}{(k+1)^2},$$

in the given range for k.

Henceforth, P will stand for the convex hull of S. Let C be the d-dimensional cube whose faces are normal to the appropriate coordinate axes, of side-length 2 and containing the unit ball B^d. On each face of C we place a $k \times k \times \cdots \times k$ $(d-1$ times) grid (so the grid points have coordinates that are integer multiples of $\frac{1}{k-1}$ and two adjacent grid points are $\frac{2}{k-1}$ apart). Note that there are fewer than $2dk^{d-1} \leq m/d$ 'grid cubes' on the union of the $2d$ faces of C. Through each grid point p, we pass a ray R from the origin. Let R intersect the unit sphere S^{d-1} at $x(R)$. For each such ray R, we choose at most d vertices of P (the convex hull of S) as follows. If the ray passes through an i-face of P, we choose $i+1$ vertices of P whose convex span intersects that ray and is contained in that i-face. Thus the set X of chosen vertices has at most m points. The convex hull of X contains the set X' of all points of the form $x(R)$ where R is a ray passing through the grid point.

Let R be any ray originating from o and suppose it intersects some face of C at a point a where a lies inside a grid cube S. Consider the triangle oab where b is any other point on the boundary of S.

$$\begin{aligned}
\sin \angle(aob) &= |ab| \cdot \frac{\sin \angle(oab)}{|ob|} \\
&\leq \frac{2\sqrt{d}}{k-1} \cdot \frac{1}{1} \leq \frac{1}{5}
\end{aligned}$$

Choose α to be

$$\alpha = \arcsin \frac{2\sqrt{d}}{k-1}.$$

Let q_0 be any point at distance $\cos\alpha$ from the origin. We show that q_0 lies in the convex hull of X'. Let R_0 be the ray from o through q_0 and suppose R_0 intersects the grid cube S_0. Let K_0 be the cone bounded by the set of rays originating from o that makes an angle of α with R_0. Hence each ray that passes through a vertex of S_0 is contained in K_0. There is a unique hyperplane H_0 containing $\partial(K_0) \cap S^{d-1}$. Note that $q_0 = R_0 \cap H_0$. Let

$$T_0 = \{x(R) : R \text{ passes through a vertex of } S_0\}$$

and

$$T_1 = \{R \cap H_0 : R \text{ passes through a vertex of } S_0\}.$$

By definition, $T_0 \subseteq X'$. Note that each point in T_0 lies on the side of H_0 not containing the origin. This means that the convex hull of X' contains the set T_1. But the convex span of the set T_1 contains the point $q_0 = R_0 \cap H_0$. This proves $r(X) \geq r(X') \geq \cos\alpha$.

$$\begin{aligned}
\cos\alpha &= (1-\sin^2\alpha)^{1/2} \\
&> 1 - \frac{\sin^2\alpha}{2} - \frac{\sin^4\alpha}{8}\sum_{i=0}^{\infty}\sin^{2i}\alpha \\
&= 1 - \frac{\sin^2\alpha}{2} - \frac{\sin^2\alpha}{8}\left[\frac{\sin^2\alpha}{1-\sin^2\alpha}\right]
\end{aligned}$$

$$\begin{aligned} &\geq 1 - \frac{97 \sin^2 \alpha}{192} \quad (\text{since } \sin^2 \alpha \leq 1/25) \\ &\geq 1 - \frac{97d}{48(k-1)^2}. \end{aligned}$$

This proves the lower bound lemma. □

4.2 Upper Bound

In this subsection, we derive an upper bound for $r_d(m)$. For this purpose, we let S be all the points on the unit sphere and then bound the largest radius of a ball contained in the convex hull of m points on the unit sphere. The convex hull of any such m points forms a polytope. The proof relies on the facts that (1) any "long" edges of this polytope bound the radius of the contained ball and (2) since the polytope has only m vertices it must have some "long" edges. The detailed calculations provide an appropriate numerical bound.

Lemma 4.2 *Let $S \subseteq \mathrm{E}^d$ be the set of all points on the surface of the d-dimensional unit ball centered at the origin o. Thus, the convex hull of S contains the unit ball B^d centered at the origin o. Then any set $X \subseteq S$ of at most m points has a residual radius*

$$r(X) \leq 1 - \frac{1}{17}\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}}, \quad \textit{for all } m \geq 3^d d^2.$$

PROOF.

The proof proceeds in two steps: We first show that for all $m > 0$ and for all $0 < \Theta < \pi/4$,

$$r(X) \leq \max\left(\cos\frac{\Theta}{2}, 1 - \frac{1 - \tan^2\Theta}{16}\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}}\right).$$

Then by an appropriate choice of the parameter Θ ($\Theta = 4\pi/53$), we obtain the claimed bound.

(1) Let X be a set of m points in E^d all lying on the surface of a unit ball and $P =$ ConvexHull(X). Let P' be the polyhedron obtained from P by triangulating the non-simplicial facets of P. Let pq be an edge of the polyhedron P'. Then

$$r(X) \leq \cos\frac{\text{“}(poq)}{2}.$$

Thus, if

$$\alpha = \max_{pq = \text{edge of } P'} \text{“}(poq),$$

is the maximum of all such angles, then

$$r(X) \leq \cos\frac{\alpha}{2}.$$

If $\alpha \geq \Theta$ then

$$r(X) \leq \cos\frac{\Theta}{2}.$$

Henceforth, we assume that $\alpha < \Theta$. Let t stand for $\tan\Theta$; thus $0 < t < 1$.

(2) Let $p \in X$ be any point, and define its *truncated cone* K_p as follows:

$$K_p = \{x\colon \text{“}(xop) \leq \alpha \text{ and } x \cdot p \leq 1\}.$$

Now, if q is an arbitrary point on the surface of the unit ball, then the line segment oq belongs to K_p, for each vertex p of some (simplicial) facet of P'. As each such simplex facet has d vertices, the collection of truncated cones cover each point in the unit ball at least d times. Thus, we see that

$$m \cdot \text{Volume}(K_p) \geq d \cdot \text{Volume(unit ball)}.$$

Let $V_d(r)$ stand for the volume of a d-dimensional ball of radius r.

$$V_d(r) = V_d(1) r^d.$$

Thus, the volume of the d-dimensional unit ball is given by

$$\begin{aligned} V_d(1) &= 2\int_0^{\pi/2} V_{d-1}(\sin\theta)\ \sin\theta\, d\theta \\ &= 2V_{d-1}(1)\int_0^{\pi/2} \sin^d\theta\, d\theta \\ &= K(d) V_{d-1}(1), \end{aligned}$$

where $K(d)$ is defined by the last equation. The volume of each K_p is given by

$$\begin{aligned} \text{Volume}(K_p) &= \int_0^1 V_{d-1}(r\tan\alpha)\, dr \\ &= V_{d-1}(\tan\alpha)\int_0^1 r^{d-1}\, dr \\ &= \frac{V_{d-1}(\tan\alpha)}{d}. \end{aligned}$$

Substituting the volumes into the preceding inequality, we get

$$m\frac{\tan^{d-1}\alpha V_{d-1}(1)}{d} \geq dK(d)V_{d-1}(1).$$

Hence,

$$1 > t = \tan\Theta > \tan\alpha \geq \left(\frac{d^2K(d)}{m}\right)^{\frac{1}{d-1}} = c(d,m),$$

where $c(d,m)$ is defined in the last equation. Using the inequality $c(d,m)^2 < t^2$, we get

$$\cos^2\alpha \leq \frac{1}{1+c(d,m)^2} \leq 1 - c(d,m)^2 + c(d,m)^4 \leq 1 - (1-t^2)c(d,m)^2.$$

Hence,

$$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1 \leq \left(1-(1-t^2)c(d,m)^2\right)^{\frac{1}{2}} \leq 1 - \frac{1-t^2}{2}c(d,m)^2$$

and

$$\cos^2 \frac{\alpha}{2} \le 1 - \frac{1-t^2}{4} c(d,m)^2.$$

Finally, we get

$$\cos \frac{\alpha}{2} \le \left(1 - \frac{1-t^2}{4} c(d,m)^2\right)^{\frac{1}{2}} \le 1 - \frac{1-t^2}{8} c(d,m)^2.$$

Hence,

$$r(X) \le 1 - \frac{1-t^2}{8} \left(\frac{d^2 K(d)}{m}\right)^{\frac{2}{d-1}}.$$

(3) Note that (e.g., [6],page 369)

$$\begin{aligned} K(d) &= 2\int_0^{\pi/2} \sin^d \theta \, d\theta \\ &= \begin{cases} \frac{(2k-1)!!}{(2k)!!}\pi, & \text{if } d = 2k = \text{even;} \\ \frac{(2k)!!}{(2k+1)!!}2, & \text{if } d = 2k+1 = \text{odd.} \end{cases} \\ &\ge 2\left(\frac{1}{2}\right)^{\frac{d-1}{2}}. \end{aligned}$$

Here $k!!$ stands for $k(k-2)(k-4)\cdots(\ell+4)(\ell+2)\ell$ (terminating in $\ell = 1$ or 2, depending on whether k is odd or even). Thus

$$r(X) \le 1 - \frac{1-\tan^2\Theta}{16} \left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}}.$$

(4) The stated bound follows with appropriate choice of the parameter Θ, as shown below: Let $m \ge 3^d d^2$; then

$$\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}} < \frac{1}{9}.$$

Choose the parameter $\Theta = 4\pi/53$, and observe that

$$\cos\frac{2\pi}{53} < 1 - \frac{1}{17\times 9} \le 1 - \frac{1}{17}\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}}.$$

Since $1 - \tan^2 \frac{4\pi}{53} > 16/17$,

$$1 - \frac{1-\tan^2(4\pi/53)}{16}\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}} \le 1 - \frac{1}{17}\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}}. \quad \square$$

If we choose $\Theta = \pi/5$ in the preceding proof, we can show that: for all $m > 0$,

$$r(X) \le 1 - \frac{15}{512}\left(\frac{d}{m}\right)^{\frac{2}{d-1}}.$$

If $m \leq d$, then

$$0 = r(X) < 1 - \frac{15}{512}\left(\frac{d}{m}\right)^{\frac{2}{d-1}}.$$

On the other hand, if $m > d$, we get the result since $\cos\frac{\pi}{10} < 1 - \frac{15}{512}$ and $1 - \tan^2\frac{\pi}{5} > 15/32$.

Summarizing lemmas 4.1 and 4.2,

Theorem 4.3 *For all* $m \geq 13^d d^{\frac{d+3}{2}}$,

$$\frac{1}{17}\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}} \leq 1 - r_d(m) \leq 3d\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}}. \quad \square$$

5 Computational Problems in the Plane

5.1 Finding m Vertices of a Convex Hull

The quantitative Steinitz's theorem poses several interesting and new problems in computational geometry. We begin with the simplest version of such problems: given a convex n-gon P whose interior contains the origin, find four vertices of P whose residual radius is maximum. In this case, we are able to give an elegant and simple linear time method. Without loss of generality, we assume that $n \geq 5$ and the interior angles at each vertex of P is less than π.

Theorem 5.1 *There is a linear time algorithm for finding a set Q of four vertices of a convex n-gon, P, such that Q has the maximum residual radius, $r(Q) = r_2(4, P)$.* $\square$

We use the following general notations. Assume that u_i's ($i = 1, \ldots, m$; $m \geq 3$) are points distinct from the origin. Let $\vec{ou_i}$ denote the ray from origin o through u_i. The notation

$$u_1 > u_2 > \cdots > u_m$$

says that the rays $\vec{ou_i}$'s are distinct and the ray $\vec{ou_i}$ is encountered before $\vec{ou_{i+1}}$ when sweeping a ray originating from o counterclockwise from $\vec{ou_1}$ to $\vec{ou_m}$. We extend this notation to the case where the u_i's are not necessarily distinct, but we still require that the rays $\vec{ou_1}$ and $\vec{ou_m}$ be distinct. For instance, we may write

$$u_1 \geq u_2 \geq u_3 \quad \text{or} \quad u_1 > u_2 \geq u_3.$$

For any point u on the boundary of the polygon P, let the *successor* $\mathrm{succ}(u)$ of u denote the vertex immediately following u when we traverse the boundary of P clockwise. If u is a vertex of P, we insist that $\mathrm{succ}(u)$ is the next vertex of P.

Our algorithm is simple to describe – its correctness is slightly harder to see. Suppose that we have four vertices u_0, u_1, u_2, u_3 of P such that there are at least three distinct vertices among them and

$$u_0 \geq u_1 \geq u_2 \geq u_3.$$

(The "$\geq$" notation here makes sense since at most one of the inequalities is non-strict.) Let $Q = u_0u_1u_2u_3$ denote the polygon formed by these vertices – so Q is a triangle or a quadrilateral.

Our goal is to repeatedly choose one of these four points, say u_i for some $i = 0, \ldots, 3$, to "advance", i.e., set u_i to $\mathrm{succ}(u_i)$, in the hope of attaining a larger residual radius. The criteria for choosing the vertex to advance depends on the following two cases. *Remark*: Here, all arithmetic on subscripts are modulo 4.

Q **is a triangle**. Suppose for some $i = 0, \ldots, 3$, u_i and u_{i+1} are *coincident*, that is $u_i = u_{i+1}$. Then we advance u_i (the 'forward vertex').

Q **is a quadrilateral.** An edge $u_{i-1}u_i$ of the quadrilateral is *limiting* if the residual circle of Q touches that edge. We then advance u_i (the 'backward vertex') where $u_{i-1}u_i$ is any such limiting edge.

We make some observations.

1. In case Q is a triangle, advancing u_i can in turn make u_i and u_{i-1} coincident, causing u_{i-1} to be advanced in the next iteration. However, there cannot be more than three consecutive iterations in which Q is a triangle. Note that a triangle Q can have non-positive residual radius.

2. In case Q is a quadrilateral and the residual radius $r(Q)$ is non-positive, the limiting edge and hence u_i is uniquely determined. After advancing u_i, provided $u_{i-1}u_i$ remains limiting, the radius $r(Q)$ will increase. This same vertex is repeatedly chosen, at least until the first time t the edge $u_{i-1}u_i$ is no longer limiting. Observe that there are two possibilities at time t: a) $r(Q)$ becomes positive, b) $r(Q)$ remains non-positive. In the latter case, the edge u_iu_{i+1} becomes limiting and we next start to advance u_{i+1}.

To complete the description of this algorithm we must initialize the four points and give the termination condition.

> **The Algorithm.** Initially, we pick any four consecutive vertices of the polygon to serve as $Q = u_0u_1u_2u_3$. We record the initial position of u_0. Then we iterate the basic step of picking and advancing an u_i, updating if necessary the largest value of $r(Q)$ encountered so far. The algorithm halts when u_0 returns to its initial position, and outputs the largest value of $r(Q)$ recorded.

It is clear that the algorithm makes at most $4n$ iterations when it halts.

For the next lemma, we need some notations. Suppose a_0, a_1, a_2, a_3 are the vertices of P that achieves the maximum residual radius $r^* = r_2(4, P)$. Without loss of generality, we may assume that all four vertices are distinct and

$$a_0 > a_1 > a_2 > a_3.$$

This partitions the vertices of the convex polygon into four non-empty *sections*, named W_0, W_1, W_2, W_3 where

$$W_0 = [a_0, a_1), \quad W_1 = [a_1, a_2), \quad W_2 = [a_2, a_3), \quad W_3 = [a_3, a_0).$$

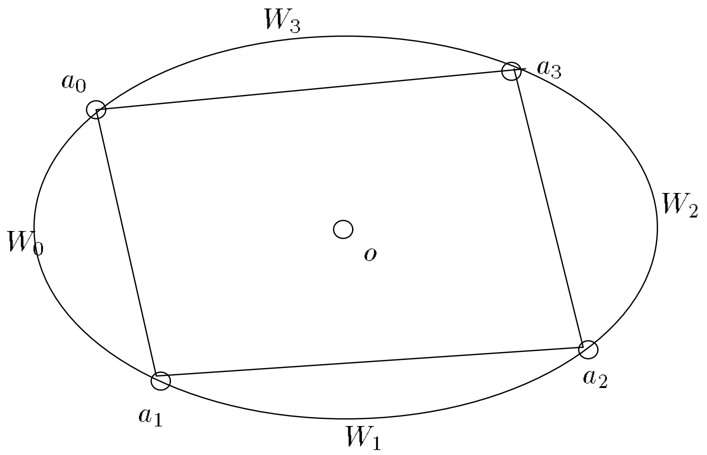

Figure 3: Q^* and four sections of the polygon P

See figure 3. The notation $[a_0, a_1)$ refers to the consecutive subsequence of vertices from a_0 counter-clockwise to (but not including) a_1. Also, let $Q^* = a_0a_1a_2a_3$.

The expression "at time t, u_j is advanced from a vertex a" means that "at time t, u_j is at vertex a and at time $t+1$ it is at $\mathrm{succ}(a)$".

Lemma 5.2 *Suppose at time t_0, some u_j ($j = 0, \ldots, 3$) is advanced from a_{i+1} of Q^*. (Hence at instant t_0+1, u_j is in section W_i.) If r^* is not yet attained by the algorithm by time t_0 then u_{j-1} is not in W_i at time t_0.*

PROOF.

Without loss of generality, assume that $i = 0 = j$, in the statement of the lemma. That is, at time t_0, u_0 is advanced from vertex a_1. By way of contradiction, suppose u_3 is in W_0 at time t_0.

There are two cases.

Case 1: Suppose Q is a triangle at time t_0. Let $t_1 < t_0$ be the last instant when Q was a quadrilateral. By a previous observation, $t_0 - t_1 \leq 3$. Note that at time t_1, for some $\ell = 0, \ldots, 3$, $u_{\ell+1}$ is advanced so that u_ℓ and $u_{\ell+1}$ became coincident. Thus u_ℓ and $u_{\ell+1}$ are adjacent at time t_1. If $r(Q) \leq 0$ at time t_1 then the origin O is on the side of the line $u_\ell u_{\ell+1}$ opposite to the other vertices of P, which is impossible. This shows that $r(Q) > 0$. Since Q is a quadrilateral, we only advance $u_\ell u_{\ell+1}$ because $u_\ell u_{\ell+1}$ is limiting. But $u_\ell u_{\ell+1}$ determines a radius greater than r^*, which leads to a contradiction.

Case 2: Suppose Q is a quadrilateral at time t_0. Then u_3u_0 is limiting at time t_0. If $r(Q)$ were positive, we deduce that $r(Q)$ is at least r^*, which is a contradiction. If $r(Q)$ were non-positive, then in order that u_3u_0 be limiting, the origin must lie to the left of the line directed from u_0 to u_3. This forces the origin to lie outside $Q*$, again leading to a contradiction. $\square$

We are now ready to show:

Lemma 5.3 *The algorithm is correct.*

PROOF.
Suppose the algorithm halts when u_0 returns to its original position b_0. Without loss of generality, assume that b_0 lies in the section $W_3 = [a_3, a_0)$. Let t_0 be the instant when u_0 is advanced from a_0 (into W_3). If $r(Q)$ has already achieved the maximum value of r^* before time t_0, then we are done. Otherwise we obtain a contradiction as follows. By the previous lemma, u_3 does not lie in W_3 at time t_0.

Claim: u_2 does not lie in $W_2 \cup W_3$ at time t_0: for if u_2 lies in W_3 then u_3 would be forced to be in W_3 as well; this is a contradiction. So it remains to show u_2 does not lie in W_2. If it does, then both u_2 and u_3 lie in W_2. Let $t_1 < t_0$ be the last time that u_3 does not lie in W_2—such an instant is well-defined. So u_3 was advanced from a_3 at time t_1. Now an application of the previous lemma again shows that $r(Q)$ would have attained the maximum value r^* before time t_1, which is a contradiction. This proves the claim.

We can repeat the argument of this claim to show that u_1 does not lie in $W_1 \cup W_2 \cup W_3$ at time t_0. Hence u_1 lies in W_0. Thus, both u_0 and u_1 lies in W_0. Again, let $t_2 < t_0$ be the last time that u_1 does not lie in W_0. Then an application of the above lemma to u_1 at time t_2 yields the contradiction. $\square$

We easily extend the above method to finding the best $m \geq 4$ vertices of the polygon P. Now we need $O(\log m)$ per iteration (using a priority queue) to find the limiting edge of the current m-gon, and the number of iterations is at most mn. This yields the following theorem.

Theorem 5.4 *For any $m \geq 4$ and $n \geq m$, there is an $O(nm \log m)$ time algorithm which on any input convex n-gon P computes the value of $r_2(m, P)$.* $\square$

5.2 Finding m Points in the General Case

The above section considers algorithms to compute $r_2(m, S)$ for the special cases, where S is the set of vertices of a convex polygon. In general, S is an arbitrary set of points in the plane, and suppose P is the subset of S consisting of all the vertices of the convex hull of S. We note that $r_2(m, S)$ is in general larger than $r_2(m, P)$. As an example, let P be the vertices of a regular pentagon and S contains, in addition to P, for each edge of the pentagon, a point in the interior of P but lying very close to the mid-point of that edge. Then $r_2(4, S) > r_2(4, P)$.

On the other hand, $r_2(m, P)$ is a reasonably good lower bound to $r_2(m, S)$. This follows from our general constructions in section 3 where the asymptotically tight lower bound for $r_2(m, S)$ is obtained by choosing points on the convex hull of S. Nevertheless, we may want to find the exact value of $r_2(m, S)$. This subsection gives such an algorithm. Again, we begin with the case $m = 4$.

Theorem 5.5 *There is an $O(n^2 \log n)$ algorithm to find a set X of four points in a set S of n points such that the residual radius of X is maximized.* $\square$

Let S be a set of points with positive residual radius. Let the points of S be assumed to be arranged by their (counter-clockwise) angular order as in the previous subsection. We want to find four points in S with the largest residual radius. For any pair u, v of distinct points, let

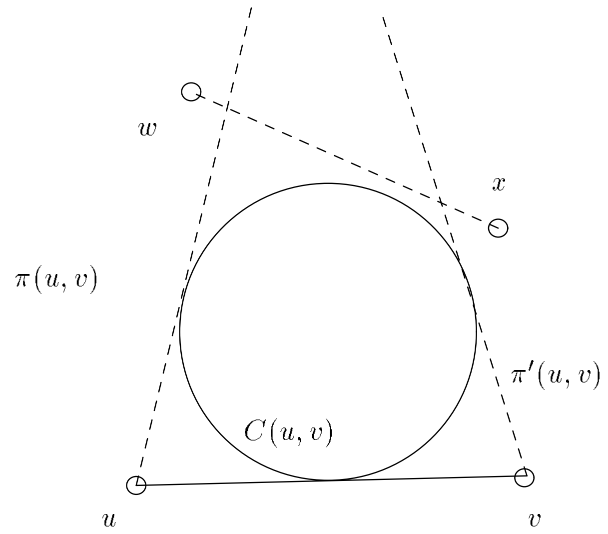

Figure 4: A quadrilateral $wuvx$ containing the circle $C(u,v)$

$C(u,v)$ denote the circle centered at the origin which has the line uv as tangent, and let $\text{rad}(u,v)$ be the radius of $C(u,v)$. If for some two points w and x of S ($w > u > v$ and $u > v > x$), $C(u,v)$ is contained in the quadrilateral $wuvx$ then the residual radius of $wuvx$ is equal to $\text{rad}(u,v)$. (Note that it is possible that $w = x$.) In the remaining portion of this subsection, we show how, for a given pair u and v, such a choice of w and x (if they exist) can be made in logarithmic time, thus providing an $O(n^2 \log n)$-time algorithm for the problem.

First, we need some notations. Let C be a circle and let u and v be points of S outside C. For a given point u, let $\bar{u}$ denote the point diametrically opposite to u (with respect to the origin o). We say u and v are *mutually C-visible* if the line segment connecting u and v does not intersect the interior of C. Also, we say v is *covered* by u (*relative to* C) if v is mutually C-visible with u and belongs to the smallest cone with its apex at u and containing C. (See figure 5). We say a point is *relevant* for C if it is outside C and not covered by any other points of S. We omit references to the circle C, if it is apparent from the context.

We have the following observations.

Lemma 5.6

1. *Let $wuvx$ be a convex quadrilateral containing $C(u,v)$. Consider two points w' and x' such that w' covers w and x' covers x. Then one of the quadrilaterals, $w'uvx'$ or $x'uvw'$, also contains $C(u,v)$.*

2. *Let $wuvx$ be a convex quadrilateral containing $C(u,v)$. Consider two relevant points w' and x' satisfying the following conditions: $\bar{u} \geq w' \geq w \geq u$; $v \geq x \geq x' \geq \bar{v}$; w' and u are mutually $C(u,v)$-visible and x' and v are mutually $C(u,v)$-visible. Then either $w'uvx'$ or $x'uvw'$ is a convex quadrilateral and contains $C(u,v)$.*

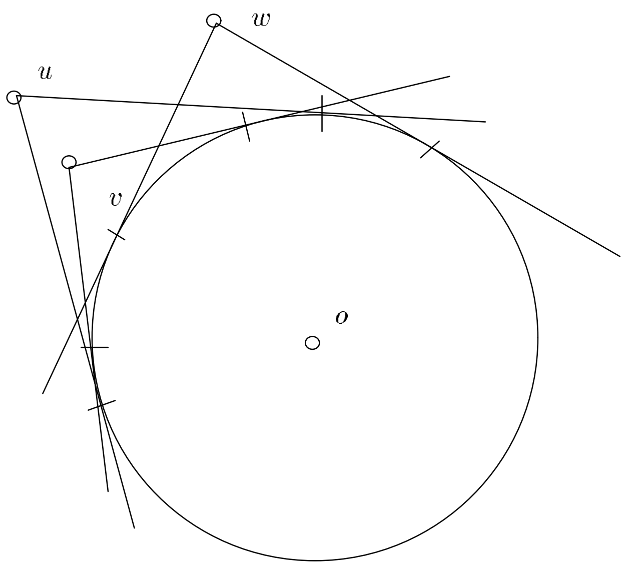

Figure 5: u covers v but not w. w covers neither u nor v

3. *Let C, C' be two concentric circles with C' being the larger of the two. If u is not relevant to C then u is also not relevant to C'.* $\square$

Let u be a point outside C and $W \subseteq S$ be the set of points w_1, w_2, ..., relevant to C such that

$$u \leq w_1 \leq w_2 \leq \cdots \leq \bar{u}.$$

Let w_j be a point in W with the largest index such that

$$\text{for all } i = 1, 2, \ldots, j, \quad u \text{ and } w_i \text{ are mutually } C\text{-visible.}$$

Then w_j is the "right-most" element in the set W, mutually visible with u. Suppose there is another element $w_k \in W$ $(k > j)$ that is mutually visible with u. Then $k \geq j + 2$ and w_{j+1} is not mutually visible with u; in this case w_{j+1} would be covered by w_k thus contradicting the hypothesis that all points of W are relevant. We say w_j is the *rightmost C-partner* of u (denoted $RP(u)$). Similarly we may define the *leftmost C-partner* of u ($LP(u)$). We conclude that $RP(u)$ and $LP(v)$ can be computed (whenever they exist) in logarithmic time, if we have a balanced search structure that contains only the points relevant to $C(u, v)$ sorted by their angular order.

Now, using the preceding lemma, we observe that if for some w and x, the quadrilateral $wuvx$ contains the circle $C(u, v)$ then so does one of the quadrilaterals, $RP(u)$ uv $LP(v)$ and $LP(v)$ uv $RP(u)$. Thus it suffices to check that (1) $C(u, v)$ is tangent to uv at some point in the segment uv and (2) $RP(u)$ and $LP(v)$ are mutually $C(u, v)$-visible.

The basic idea is to put all vertices u, and also all unordered pairs $\{u, v\}$ of points in S into a single priority queue. We use the Euclidean distance between u and the origin o as priority of u, and the value of $\text{rad}(u, v)$ as priority of $\{u, v\}$. We may omit all $\{u, v\}$'s with $\text{rad}(u, v) = 0$

and begin the processing of the queue by successively extracting items with the smallest priority. Note that we could assume that the first item extracted is a pair $\{u, v\}$.

For the first pair $\{u, v\}$ extracted from the queue, we need to initialize a data structure to store the points of S relevant to $C(u, v)$ according to their angular order. We may break the circular ordering into a linear ordering at some arbitrary breakpoint. We store these points as a linear ordering in the leaves of a balanced binary tree T.

In the general step, suppose we extract from the priority queue either a pair $\{u, v\}$ or vertex u. There are two cases.

- Case 1: A pair $\{u, v\}$ is extracted and $C(u, v)$ touches uv at some point z in the segment uv. Then assuming $u \geq z \geq v$, we use the search tree T to determine the rightmost $C(u, v)$-partner w of u and the leftmost $C(u, v)$-partner x of v and check if w and x are mutually $C(u, v)$-visible. If so, we have found a larger residual radius.

- Case 2: (a) A pair $\{u, v\}$ is extracted and $C(u, v)$ touches uv at some point outside the segment uv. Assume that of the two points u, v, the point u is the closer to o. Then we note that for subsequent (larger radii) circles C, the vertex v covers u relative to C. Hence we can delete the element u from the data structure T. (b) Similarly if u is extracted, we can delete u from T.

Clearly each operation takes $O(\log n)$ time. Since there are $O(n^2)$ elements in the queue, the overall complexity is $O(n^2 \log n)$. Note that at any instant when a pair $\{u, v\}$ is being considered, only relevant points of S are left in T. More precisely, any point in the interior of $C(u, v)$ or covered by some other point would have already been deleted. (A point in the interior of $C(u, v)$ is at a smaller distance from o than $\text{rad}(u, v)$, and if v' covers u' then $\text{rad}(u', v') \leq \text{rad}(u, v)$; $u'v'$ is touched by $C(u', v')$ outside the segment $u'v'$ and u' is closer to o than v'.) Thus it is clear that the algorithm is correct.

Again the method generalizes to finding any number m of points that has the best residual radius. This yields the following theorem.

Theorem 5.7 *For any $m \geq 4$ and $n \geq m$, there is an $O(n^2 m \log n)$ time algorithm which on any input set S of n points in the plane, computes the value of $r_2(m, S)$.* □

We note that a faster algorithm is possible if we are willing to settle for a good (to within a factor $\geq (1 - \epsilon)$, $0 < \epsilon < 1$) approximation of $r_2(m, S)$, for all $m \geq 4$. Specifically, we can determine if $r_2(m, S)$ is $<$ or $\geq$ a fixed value r in time $O(nm \log n)$. (In $O(n \log n)$ time we can determine the set of points that are relevant for a circle of radius r. For each relevant point u, in $O(m \log n)$ time, we can determine if there exist $\leq (m - 1)$ other additional points such that the set containing u together with these points has a residual radius of r or larger.) To begin with, we choose the best set of *four* vertices on the convex hull of S, and call its residual radius r_0. Using an $O(n \log n)$ time convex hull algorithm, and the algorithm of the previous subsection, we guarantee that this step takes no more than $O(n \log n)$ time. Thus

$$0.30 < \frac{r_0}{r_2(m, S)} \leq 1.$$

Using $k = O(\log(1/\epsilon))$ comparisons, we can perform a binary search to improve the approximation to

$$1 - \epsilon \leq \frac{r_k}{r_2(m, S)} \leq 1.$$

Thus, the resulting algorithm computes a good approximation (with a relative error of ϵ) in time $O(nm \log n \log(1/\epsilon))$.

6 Computational Problems in Higher Dimensions

In this section, we study the following algorithmic problem:

> Given a set S of n points in d-dimensional Euclidean space, whose residual radius $r(S)$ is positive, find a subset $X \subseteq S$ of at most m points such that the following inequality holds:
>
> $$\frac{r(X)}{r(S)} \geq \tilde{r}_d(m) = 1 - 3d\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}}.$$
>
> Here m and n are assumed to be sufficiently large, i.e. $n \geq m \geq 13^d d^{\frac{d+3}{2}}$.

We shall not discuss the more general "optimization" problem of finding a subset of m points that maximizes the preceding ratio, for two reasons: firstly, for large m, the approximate solution provides a reasonably good answer; secondly, any hope for finding such a set in time polynomial in both d and n seems rather dim. While an investigation of this optimization problem is called for, we simply leave it as an open problem.

Returning to the stated problem, we see that this problem can be solved by essentially following the ideas outlined in lemma 4.1: We first choose a set X' of at most m/d points on the surface of the unit ball such that the residual radius of X' is no smaller than $\tilde{r}_d(m)$. We can then determine a set $X \subseteq S$ of at most m points such that for some $\lambda_{\min} \geq r(S)$, the convex hull of X contains the set of points

$$\lambda_{\min} X' = \{\lambda_{\min}\, q \colon q \in X'\}.$$

Thus

$$r(X) \geq r(\lambda_{\min} X') \geq \lambda_{\min} \tilde{r}_d(m) \geq r(S)\tilde{r}_d(m).$$

The points of X' are chosen as follows: Let C be the d-dimensional cube comprising the points $(x_1, \ldots, x_d)$ with $|x_i| \leq 1$ for $i = 1, \ldots, d$. On each face of C, we place a $k \times k \times \cdots \times k$ ($(d-1)$ times) grid, with k taking the value

$$\left\lfloor \left(\frac{m}{2d^2}\right)^{\frac{1}{(d-1)}} \right\rfloor.$$

Let

$$X' = \left\{\vec{op} \cap \mathrm{S}^{d-1} \colon p \text{ is a grid point}\right\}.$$

Thus $|X'| \le 2dk^{d-1} \le m/d$. For each $q \in X'$, we determine an appropriate set $S_q \subseteq S$ of at most d points such that

$$\vec{oq} \cap \partial\,(\mathrm{ConvexHull}(S)) \in \mathrm{ConvexHull}(S_q);$$

thus for some λ_q,

$$\lambda_q q \in \mathrm{ConvexHull}(S_q).$$

Let X be

$$X = \bigcup_{q \in X'} S_q,$$

with $\lambda_{\min}$ taking the value $\min_{q \in X'} \lambda_q$. Evidently, $\lambda_{\min} \ge r(S)$.

Note that $|X| \le m$, and

$$\lambda_{\min} X' \subseteq \mathrm{ConvexHull}(X).$$

This demonstrates the correctness of the algorithm, since we know that the residual radius of X' is bounded from below by $\tilde{r}_d(m)$ (see the proof of lemma 4.1).

In order to complete the algorithm, we show how to efficiently compute the set S_q (for any point q) using the following linear programming formulation. Let $S = \{p_1, p_2, \ldots, p_n\}$. Without loss of generality, we assume that the points of S are in *general position*, i.e., *at most d points of S may lie on any* $(d-1)$ *dimensional hyperplane*. If not, the original points of S may be perturbed using generic perturbation methods (see, for example, [16]); the following discussions still apply *mutatis mutandis*. Define the $d \times n$ matrix $\mathbf{A}$ whose j^{th} column consists of the coordinates of the point p_j. Corresponding to the point q, define a column d-vector $\mathbf{b}$. The linear programming problem (LP) is given as follows:

- Given: A $d \times n$ matrix $\mathbf{A}$ and a column d-vector $\mathbf{b}$.
- Solve:

$$\begin{array}{lrcl} \text{minimize} & -\lambda & & \\ \text{subject to} & \mathbf{A}\mathbf{x} & = & \lambda \mathbf{b} \\ & \mathbf{e}^{\mathrm{T}}\mathbf{x} & = & 1 \\ & \mathbf{x} & \ge & \mathbf{0} \\ & \lambda & \ge & 0, \end{array}$$

  where $\mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{T}}$, $\mathbf{e} = (1, \ldots, 1)^{\mathrm{T}}$ and $\mathbf{0} = (0, \ldots, 0)^{\mathrm{T}}$ are column n-vectors.

Let $\mathbf{x}^*$, λ^* be an optimal solution of (LP). Then $\lambda^* > 0$ is the maximum value of λ such that

$$\lambda^* q = \sum_{i=1}^{n} x_i^* p_i,$$

with $\sum_{i=1}^{n} x_i^* = 1$, and $x_i^* \ge 0$.

Now consider the following *dual* of the (LP), which will be referred to as (DLP):

$$\begin{array}{lrcl} \text{maximize} & y_{d+1} & & \\ \text{subject to} & a_{1,1}y_1 + \cdots + a_{d,1}y_d + y_{d+1} & \le & 0 \\ & a_{1,2}y_1 + \cdots + a_{d,2}y_d + y_{d+1} & \le & 0 \\ & & \vdots & \vdots \\ & a_{1,n}y_1 + \cdots + a_{d,n}y_d + y_{d+1} & \le & 0 \\ & -b_1y_1 - \cdots - b_dy_d & \le & -1 \end{array}$$

This problem can be solved in $O(3^{d^2}n)$ time by using Clarkson-Dyer's improvement on Megiddo's multidimensional search technique [4,5,7]. Let us now see how to recover the solution to the original problem.

Clearly both (LP) and (DLP) have optimal solutions. Let an optimal solution for (DLP) be

$$\mathbf{y}^* = (y_1^*, \ldots, y_d^*, y_{d+1}^*).$$

Let $I_q \subseteq \{1..n\}$ be the set of all the indices j such that

$$\mathbf{a}_j \cdot \mathbf{y}^* = a_{1,j}y_1^* + \cdots + a_{d,j}y_d^* + y_{d+1}^* = 0.$$

where $\mathbf{a}_j = (a_{1,j}, \cdots, a_{d,j}, 1)^{\mathrm{T}}$. By the Complementary Slackness Theorem (see [3]), this implies that for all $i = 1, \ldots, n$, if $x_i^* > 0$ then $i \in I_q$. By virtue of our non-degeneracy hypothesis about the points of S, we see that $|I_q| \leq d$. We now claim that $S_q = \{p_j : j \in I_q\}$ can serve as a desired solution. Clearly, $S_q \subseteq S$, has at most d points and

$$\overrightarrow{oq} \cap \partial(\mathrm{ConvexHull}(S)) \in \mathrm{ConvexHull}(S_q)).$$

Note that even if the original set had been perturbed (by a sufficiently small amount) the set S_q chosen from the unperturbed set S still provides the desired solution.

To summarize:

Theorem 6.1 *For $n \geq m \geq 13^d d^{(d+3)/2}$, we can find a set X of at most m points from an input set S of n points such that*

$$\frac{r(X)}{r(S)} \geq \tilde{r}_d(m) = 1 - 3d\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}},$$

in time $O(3^{d^2}mn)$. □

7 Final Remarks

It is natural to seek improved forms of Steinitz's theorem for certain subsets $S \subseteq \mathrm{E}^d$. In other words, if k is any number (between $d+1$ and $2d$), we want to characterize those subsets $S \subseteq \mathrm{E}^d$ whose residual radius is positive and are such that S contains a subset X of at most k points, where X has a positive residual radius. For instance, in the plane:

Lemma 7.1 *Let $S \subseteq \mathrm{E}^2$ be any set with positive residual radius. Then there is a subset of three points in S with positive residual radius if and only if S is not contained in two lines through the origin.*

We omit the easy proof. It would be interesting to develop an appropriate quantitative forms of this lemma. We see that an obvious quantitative version for this lemma fails. That is, there does not exist a constant $0 < \alpha < 1$ with the following property:

> Suppose the residual radius of $S \subseteq \mathrm{E}^2$ is at least one and S does not lie in two lines through the origin. Then there exists three points in S whose residual radius is at least α.

To see this, consider the set $S = \{A, B, C, D, E\}$ where $A = (0, 1), B = (1, 0), C = (-1, 0), D = (1, -L)$ and $E = (-1, -L)$ for $L = L(\alpha) > 0$ sufficiently large. Then no subset of S with three points has residual radius at least α.

Yet another area of research that calls for further investigation arises from the observation that the torque and force dimensions are really non-comparable. We want a notion of grasp-efficiency that can take this into account. A related issue is that the current approach depends on the origin of the reference frame in which the torques are measureed. Is there an origin-independent approach to efficiency and other metrics of a grasp?

8 Acknowledgment

We are pleased to acknowledge some helpful discussions with Professor S. Rao Kosaraju of Johns Hopkins University and Professor Richard Pollack of the Courant Institute. We also thank an anonymous referee who pointed out a calculation error in the lemma 4.2.

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