On Immobility

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1 Introduction

The problem of immobilizing an object consists of finding a set of "fingers" whose contact with the object prevents its motion. Various models of immobilization exist and are closely related.

The area of dexterous manipulation studies the problems of grasping, fine-manipulation and control with multi-fingered robot hands (see [Asa79, JWKB84, Oka82, SR81, TB62]. These hands usually consist of a small number (four or five) of articulated fingers, possibly with a palmar surface.

The science and technology of dexterous manipulation, however, are still in their infancy. Many of the grasping and control algorithms for manipulation implicitly assume fairly well-controlled and well-modeled environments: for instance, the algorithms developed by [TAP87] is strongly dependent on availability of gravitational force. In an attempt to develop schemes that work in more general situations, researchers have begun to study the underlying mathematics; these results are somewhat theoretical in nature and assume a rather idealized hand.

In manufacturing, fixtures and other workholding devices need to be placed so as to secure the stock to the tool base and withstand high cutting loads while machining takes place [HW90].

Similar problems can be posed in a geometric setting. A set of points is placed in $\mathbb{R}$ and the points are forbidden to penetrate the interior of the object. Questions such as how many points are needed and how should they be placed arise. The techniques used to solve these problems bear much similarity to those used in grasping theory.

Here we shall attempt to survey some of the results, investigate the techniques employed and pose some of the open problems.

This paper is organized as follows: section two describes the basic theory and terminology related to grasping. Section three discusses the techniques employed in analysis and synthesis of positive grips. Section four introduces the concept of grasp efficiency and discusses attempts to obtain optimal grips under various definitions of optimality. Section five mentions some problems related to finite immobility. A concluding section describing the future research follows.

2 Grasping: Terminology and Theory

Consider an idealized robot hand, consisting of several independently movable force-sensing fingers; this hand is used to grasp a rigid object B. Furthermore, we make the following simplifying assumptions:

- (*Smooth Body*) B is a full-bodied (i.e. no internal holes) compact subset of the Euclidean 3-space. Furthermore, B has a piece-wise smooth boundary ∂B.

- (*Point Contact*) For each finger-contact on the body, we may associate a nominal point of contact, $\mathbf{p} \in \partial B$. We let ∂B^* denote the set of points $\mathbf{p} \in \partial B$ such that the direction $\mathbf{n}(\mathbf{p})$ normal to ∂B at $\mathbf{p}$ is well-defined; by convention, we pick $\mathbf{n}(\mathbf{p})$ to be the unit normal pointing into the interior of B.

  For each such point $\mathbf{p}$, we can define a wrench system $\{\Gamma^{(1)}(\mathbf{p}), \Gamma^{(2)}(\mathbf{p}), \ldots, \Gamma^{(m)}(\mathbf{p})\}$, $(0 \leq m \leq 6)$, where the number and screw-axes of the wrench system depend on the contact type. Some of these wrenches can be *bisense* (i.e. can act in either sense) and the remaining wrenches, *unisense*. (For a discussion of screw theory, and in particular, wrenches and twists, see [Hun78] and [Ohw80].)

- (*Compliance*) We will consider the case when the fingers are stiff—the force/torques applied at the fingers are generated by some actuators whose mechanics need not concern us.

Many interesting special cases occur, depending on how we model the *static friction* and the *stiction* between the fingers and the body B. In the case, where the contacts are frictionless, a finger can only apply force $\mathbf{f}$ on the body in the direction $\mathbf{n}(\mathbf{p})$ at the point $\mathbf{p}$. Also if the fingers are non-sticky, then the force $\mathbf{f}$ has a non-negative magnitude, $f = \mathbf{f} \cdot \mathbf{n}(\mathbf{p}) \geq 0$. Such grips are also known as 'positive grips'. In this case, the wrench system associated with each point is:

$$\Gamma(\mathbf{p}) = \{[\mathbf{n}(\mathbf{p}), \mathbf{p} \times \mathbf{n}(\mathbf{p})]\}$$

Thus, corresponding to a set of finger-contacts, we have a system of n wrenches,

$$\{\mathbf{w}_1, \ldots, \mathbf{w}_k, \mathbf{w}_{k+1}, \ldots, \mathbf{w}_n\},$$

the first k of which are bisense and the remaining last $n - k$ of the wrenches are unisense. Let us assume that the magnitudes of these wrenches are given by the scalars f_i's

$$\{f_1, \ldots, f_k, f_{k+1}, \ldots, f_n\},$$

where $f_1, \ldots, f_k \in \mathbb{R}$ and $f_{k+1}, \ldots, f_n \in \mathbb{R}_+$, and not all the magnitudes are zero. We call such a system of wrenches and the wrench-magnitudes, a *grip*, G, and say that this grip G generates an external wrench $\mathbf{w} = [F_x, F_y, F_z, \tau_x, \tau_y, \tau_z] \in \mathbb{R}^6$, if

$$\mathbf{w} = \sum_{i=1}^{n} f_i \ \mathbf{w}_i.$$

Next, we consider the concept of a *closure grasp*: A system of wrenches $\mathbf{w}_1, \ldots, \mathbf{w}_n$ (as before) is said to constitute a *force/torque closure grasp* if and only if any arbitrary external wrench can be generated by varying the magnitudes of the wrenches (subject to the constraints imposed by the senses of the wrenches). A necessary and sufficient condition for a closure grasp is that the (module) sum of the linear space spanned by the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_k$ and the positive space spanned by the vectors $\mathbf{w}_{k+1}, \ldots, \mathbf{w}_n$ is the entire $\mathbb{R}^6$:

$$\text{lin } (\mathbf{w}_1, \ldots, \mathbf{w}_k) + \text{pos } (\mathbf{w}_{k+1}, \ldots, \mathbf{w}_n) = \mathbb{R}^6.$$

Let us denote, by L, the linear space lin $(\mathbf{w}_1, \ldots, \mathbf{w}_k)$, and, by $L^{\perp}$, the orthogonal complement of L in $\mathbb{R}^6$. Let π be the linear projection function of $\mathbb{R}^6$ onto $L^{\perp}$ whose kernel is L. Then it can be shown that a necessary and sufficient condition for a closure grasp is

$$\text{lin } (\mathbf{w}_1, \ldots, \mathbf{w}_k) + \text{pos } (\pi\mathbf{w}_{k+1}, \ldots, \pi\mathbf{w}_n) = \mathbb{R}^6.$$

The above equation in turn is equivalent to the following conditions:

$$\mathbf{0} \in \text{int conv } (\pi\mathbf{w}_{k+1}, \ldots, \pi\mathbf{w}_n)$$

in $L^{\perp}$. Here, if $k = 0$ (i.e. positive grip) then the above condition reduces to the following:

$$\mathbf{0} \in \text{int conv } (\mathbf{w}_1, \ldots, \mathbf{w}_n).$$

At this point, we emphasize that the above formulation has turned a problem in mechanics into a purely geometric problem, now amenable to many interesting techniques in convexity theory and computational and combinatorial geometry. This observation has been most fruitfully exploited in the work of [MSS87] and [MS89].

Form closure

Another approach to immobility is that first introduced in [Lak78] and further examined in [Ohw80].

Each wrench associated with a contact point reduces the set of possible twists that the body can undergo. The remaining twists are exactly those whose virtual coefficient with any wrench is non-negative, since otherwise the virtual work done by such twists would be negative.

A set of twists associated with the contacts constitutes a *form closure* if any arbitrary twist d is resisted by the set of contacts, i.e. if d is non-reciprocal to some $\mathbf{w}_i$ $(i = 1 \ldots k)$ or contrary to some $\mathbf{w}_j$ $(j = k+1 \ldots n)$. This is in turn equivalent to

$$\forall d' \in \mathbb{R}^6 \;\; \mathbf{w}_i \cdot d' = 0 \quad 1 \leq i \leq k \;\; \text{implies}$$

$$\mathbf{w}_j \cdot d' = \pi\mathbf{w}_j \cdot d' < 0 \;\; \text{for some } j, \;\; k+1 \leq j \leq n.$$

The last equation is equivalent to

$$\pi\mathbf{w}_j \cdot d' \geq 0 \qquad k+1 \leq j \leq n \;\; \text{is unsolvable}$$

which can be shown [Chv83] to be equivalent to

$$\text{pos}(\pi\mathbf{w}_{k+1}, \ldots, \pi\mathbf{w}_n) = L^{\perp}$$

Hence form closure, also referred to as *zero total freedom*, and force closure are equivalent.

Finite immobility

A different concept of immobility, of more geometric flavor, has recently been introduced by Kuperberg [Kup90], and discussed in [O'R90b] and [CSU90]. A set of points I in $\mathbb{R}^3$ is said to *immobilize* an object B if any rigid motion of the object causes at least one point of I to penetrate the interior of B.

Czyzowicz, Stojmenovic and Urrutia [CSU90] establish that four points suffice to immobilize any object in the plane, and $2d$ points are sufficient in $\mathbb{R}$ using techniques such as finding the largest sphere inscribed inside the object. The arguments bear some similarity to those used in [BFG85] and [MNP90] to show analogous results about the number of fingers required for closure grasps (see below.)

In fact, the number of points necessary to finitely immobilize an object is no greater that the number of fingers required for a form closure grasp of this object.

In the plane, this can be seen as follows. It is well known that any planar displacement can be regarded as a rotation about a center of rotation (potentially at infinity for translations.) Let $\mathbf{p}$ be the contact point between a finger and the object B, let $\mathbf{n}(\mathbf{p})$ be the outward pointing normal to B at $\mathbf{p}$ and $\mathbf{n}'(\mathbf{p})$ be $\mathbf{n}(\mathbf{p})$ rotated counterclockwise by $\pi/2$. Then the set of rotation centers around which clockwise rotations are permitted by p is $\{\mathbf{x} \in \mathbb{R}^{\nvDash} : \mathbf{n}'(\mathbf{p}) \cdot \mathbf{px} \geq \mathbf{0}\}$.

Under the finite immobility model, and for immobilizing points only on regular points of ∂B, the set of rotation centers lying on the relatively open half-line h from p towards the interior of B is not included in the set H' of centers of allowed rotations, i.e. $H' = H \setminus h$. Therefore the finite model is less restrictive in the plane, and the same situation occurs in $\mathbb{R}^{\nVdash}$. In addition, allowing immobilizing points on singular points of ∂B (i.e. vertices and edges, etc.) may reduce the number of fingers required.

We can summarize the relationship between the various definitions of immobility as follows.

$$\text{Force Closure} \iff \text{Form Closure} \Longrightarrow \text{Finite Immobility}$$

where *Def A* $\Longrightarrow$ *Def B* means *if a body is immobilized by n fingers under Def A then it can be immobilized by n or fewer fingers under Def B.*

3 The case of a Positive Grip

In order to obtain a particular grasp on an object, it must be determined if that grasp is achievable. It is for this reason, researchers have studied the question of how many fingers (wrenches) are required to obtain certain grasps on the object.

Reuleaux and Somoff determined that the closure grasp of a two dimensional object requires at least *four* forces and of a three dimensional object requires at least *seven*, where the forces are normal to the surface of the object.

Mishra, Schwartz, and Sharir [MSS87] gave the most general bounds on the number of fingers in the case of a *positive grip*. In this section, we briefly describe their techniques.

Recall that a non-frictional grip is called a *positive grip*. Note that, in this case, the fingers are assumed to be point fingers, a finger can only apply a force on the object along the surface-normal at the point of contact, directed inward. In this situation, we have a *wrench map*, Γ, mapping ∂B^* into the six-dimensional *wrench space* $\mathbb{R}^{\ntrianglelefteq}$ as follows:

$$\Gamma \colon \partial B^* \to \mathbb{R}^{\ntrianglelefteq} \; : \mathbf{p} \mapsto [\mathbf{n}(\mathbf{p}), \mathbf{p} \times \mathbf{n}(\mathbf{p})].$$

Essentially, Γ maps $\mathbf{p}$ to the point $\Gamma(\mathbf{p})$ in the wrench space that represents the effects of applying a unit force at $\mathbf{p}$ in the direction $\mathbf{n}(\mathbf{p})$.

Now, given an arbitrary compact rigid object B whose piece-wise smooth surface ∂B is not a "surface of revolution," the following can be shown.

B can be held with a closure grasp by a positive grip of at most twelve fingers.

The assumption that ∂B is not a surface of revolution is essential, since, in the absence of friction, no matter how many fingers are employed the object can always be rotated about its axis of symmetry.

By virtue of the discussion of the previous section, we only need to show that

$$\Big(\exists\, \mathbf{p}_1, \ldots, \mathbf{p}_k \in \partial B^*,\ k \leq 12\Big)\ \Big[\mathbf{0} \in \text{int conv}\ \Big(\Gamma(\mathbf{p}_1), \ldots, \Gamma(\mathbf{p}_k)\Big)\Big].$$

The proof is in two steps:

(1) We claim that

$$\mathbf{0} \in \text{int conv}\ \Gamma(\partial B^*).$$

Using Gauss' divergence theorem, we see that

$$\left[\int_{\partial B^*} \mathbf{n}(\mathbf{p})\, dS,\ \int_{\partial B^*} (\mathbf{p} \times \mathbf{n}(\mathbf{p}))\, dS\right] = \mathbf{0},$$

i.e.

$$\mathbf{0} \in \text{conv}\ \Gamma(\partial B^*).$$

Thus, it remains to show that the origin is indeed an interior point of the convex hull; assume to the contrary. Then it can be shown that there is a nonzero vector $\mathbf{g} = [F, \tau] \in \mathbb{R}^6$ orthogonal to the linear subspace spanned by $\Gamma(\partial B^*)$. Thus, for each function $c(\mathbf{p})$ we have

$$\int_{\partial B^*} c(\mathbf{p})\ \left[(F + \tau \times \mathbf{p}) \cdot \mathbf{n}(\mathbf{p})\right]\, dS = 0.$$

In particular, substitute the function $c(\mathbf{p}) = (F + \tau \times \mathbf{p}) \cdot \mathbf{n}(\mathbf{p})$ into the last equation, to deduce that $(F + \tau \times \mathbf{p}) \cdot \mathbf{n}(\mathbf{p})$ must be identically 0 over ∂B^*. But, this is possible only in the case when the surface of B is a surface of revolution.

(2) The rest follows by an application of the following theorem from combinatorial geometry:

Steinitz's Theorem. [Ste]
If $X \subseteq \mathbb{R}$ and $\mathbf{p} \in$ int conv X, then $\mathbf{p} \in$ int conv Y for some $Y \subseteq X$ with $|Y| \leq 2d$.

Our main result then follows from Steinitz's Theorem, once we identify X with $\Gamma(\partial B^*)$, and Y with the set $\{\Gamma(\mathbf{p}_1), \ldots, \Gamma(\mathbf{p}_k)\}$, with $k \leq 2 \times 6 = 12$.

Other results

For certain types of object, it can be shown that 7 fingers are sufficient thus closing the gap between the bounds for those objects.

For convex polyhedral objects with n vertices, Meyer [Mey90] provides an $O(n^{3/2}\sqrt{(}\log n))$ algorithm to find a seven finger closure grasp. The construction proceeds as follows.

First, the two vertices which are endpoints of a diameter of the object are found. Then near each of those vertices, a small sphere is found so that it touches three of more faces adjacent to the corresponding diameter vertex.

Let the centers of the spheres be called $\mathbf{c}_l$ and $\mathbf{c}_h$ respectively, and assume $\mathbf{c}_l$ is the origin and $\mathbf{c}_h$ has coordinates $(0, 0, 1)$. Also let C_l (resp. C_h) be the convex cone generated by the outward pointing normals of the faces touching the sphere with center $\mathbf{c}_l$ (resp. $\mathbf{c}_h$). They are the convex hull of their extreme rays. It is shown that $\mathbf{c}_l$ and $\mathbf{c}_h$ can be chosen so that $\mathbf{c}_h - \mathbf{c}_l \in \text{int } C_h \cap \text{int } (-C_l)$. Then using a planar version of Steinitz' theorem applied to the intersection of a convex cone and a plane, three normals $\mathbf{n}(\mathbf{p}_i)$, $1 \leq i \leq 3$ (resp. $4 \leq i \leq 6$) can be chosen out of the extreme rays of C_l (resp. C_h) such that $\mathbf{c}_h - \mathbf{c}_l \in \text{int pos}\,(\mathbf{n}(\mathbf{p}_1), \mathbf{n}(\mathbf{p}_2), \mathbf{n}(\mathbf{p}_3))$, (resp. $\mathbf{c}_l - \mathbf{c}_h \in \text{int pos}\,(\mathbf{n}(\mathbf{p}_4), \mathbf{n}(\mathbf{p}_5), \mathbf{n}(\mathbf{p}_6))$. Some perturbation of $\mathbf{c}_h$ might be required in degenerate case.

Thus the six normals satisfy $\sum_1^3 \alpha_i \mathbf{n}_i + \sum_4^6 \alpha_i \mathbf{n}_i = \mathbf{0}$, for some $\alpha_i > 0$. Now consider the torque space. Since the forces at $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ (resp. $\mathbf{p}_4$, $\mathbf{p}_5$, $\mathbf{p}_6$) are concurrent at $\mathbf{c}_l = \mathbf{0}$, (resp. $\mathbf{c}_h$), we have

$$\sum_{i=1}^{3} \alpha_i (\mathbf{p}_i \times \mathbf{n}_i) = \mathbf{0}$$

and

$$\sum_{i=4}^{6} \alpha_i (\mathbf{p}_i \times \mathbf{n}_i) = \sum_{i=4}^{6} \alpha_i (\mathbf{c}_h \times \mathbf{n}_i) = \mathbf{c}_h \times \sum_{i=4}^{6} \alpha_i \mathbf{n}_i = \mathbf{c}_h \times (\mathbf{c}_h - \mathbf{c}_l) = \mathbf{0}.$$

Hence $\sum_1^6 \alpha_i \Gamma(\mathbf{p}_i) = \mathbf{0}$, $\alpha_i > 0$ and $\{\mathbf{p}_i, \ldots, \mathbf{p}_6\}$ constitutes an equilibrium grasp.

To obtain a closure grasp, one of the points, say $\mathbf{p}_6$ is replaced by $\mathbf{p}_6 + \mathbf{u}$ and $\mathbf{p}_6 - \mathbf{u}$. Since $\mathbf{u}$ has a two dimensional set of possibilities, it can be shown that there is a $\mathbf{u}$ such that $\Gamma(\mathbf{p}_1)$, $\Gamma(\mathbf{p}_2), \ldots, \Gamma(\mathbf{p}_5)$, $\Gamma(\mathbf{p}_6 + \mathbf{u})$ are linearly independent. Then since $\frac{1}{2}\Gamma(\mathbf{p}_6 + \mathbf{u}) + \frac{1}{2}\Gamma(\mathbf{p}_6 - \mathbf{u}) = \Gamma(\mathbf{p}_6)$, we have

$$\sum_{1}^{5} \alpha_i \Gamma(\mathbf{p}_i) + \frac{\alpha_6}{2}\Gamma(\mathbf{p}_6 + \mathbf{u}) + \frac{\alpha_6}{2}\Gamma(\mathbf{p}_6 u \mathbf{u}) = \mathbf{0}.$$

By a theorem on non-negative basis [GT56] the seven wrenches positively span $\mathbb{R}^{\not 6}$ and we have a force/torque closure grasp.

Markenscoff, Ni and Papadimitriou [MNP90] also have a similar result for more general objects without rotation symmetry. Their proof also uses a perturbation argument but is sketchy and the statement of their main theorem appears incomplete.

Algorithmic Techniques

Given the bounds on the number of fingers, and a polyhedron P of n faces, we wish to find a set of fingers and force targets that will produce a force-torque closure grasp.

Meyer's algorithm has been briefly described in the previous section. The complexity of the algorithm is due mainly to finding the diameter of the convex polyhedron. It would be interesting to determine whether another pair of vertices that is not so hard to find might suffice.

Mishra, Schwartz and Sharir [MSS87] have a more algebraic algorithm for any polyhedral object which has two phases. The first phase finds a set of $O(n)$ fingers forming a force-torque closure grasp. The second then reduces the number of fingers to a constant number.

In the first phase, three triangular regions, each on a different face of P and adjacent to the same vertex are found and each is further subdivided into three triangular regions whose centroids are not collinear. Let $N = O(n)$ be the number of triangular regions on ∂P. Let $\mathbf{p}_i$ be the centroid of region i and α_i be its area. Then it follows from the argument above involving Gauss' divergence theorem that

$$\sum_{1}^{N} \alpha_i \Gamma(\mathbf{p}_i) = 0. \tag{1}$$

We therefore have a positive grip with force targets α_i, which can clearly be found in $O(n)$ time.

Assuming $\mathbf{p}_i$, $1 \leq i \leq 9$, belong to the three subdivided faces, it is shown for any wrench $\mathbf{w}$ that $\mathbf{w} \in \text{Lin}\left(\Gamma(\mathbf{p}_1), \ldots, \Gamma(\mathbf{p}_9)\right)$. Hence

$$\sum_{1}^{9} \beta_i \Gamma(\mathbf{p}_i) = \mathbf{w} \tag{2}$$

for some β_i. Now since $\alpha_i > 0$, we can add a sufficiently large multiple of (1) to (2) to obtain $\mathbf{w} \in \text{pos}\left(\Gamma(\mathbf{p}_i), \ldots, \Gamma(\mathbf{p}_N)\right)$ as required for a closure grasp.

Phase two is accomplished by eliminating fingers until the required number k given by the bounds is reached.

We start with the equation

$$\alpha_1 \Gamma(\mathbf{p}_1) + \cdots + \alpha_N \Gamma(\mathbf{p}_N) = -(\Gamma(\mathbf{p}_1') + \cdots + \Gamma(\mathbf{p}_6')) = \alpha \mathbf{v},$$

where $\{\mathbf{p}_1', \ldots, \mathbf{p}_6'\} \subseteq \{\mathbf{p}_1, \ldots, \mathbf{p}_9\}$ and $\text{Lin}\left(\Gamma(\mathbf{p}_1'), \ldots, \Gamma(\mathbf{p}_6')\right) = \mathbb{R}^6$. Initially, $\alpha = 1$. Each phase of elimination proceeds as follows: Assume $\Gamma(\mathbf{p}_1), \ldots, \Gamma(\mathbf{p}_6)$ are linearly independent. Then there exist $\beta_1, \ldots, \beta_6$ such that $\beta_1 \Gamma(\mathbf{p}_1) + \cdots + \beta_6 \Gamma(\mathbf{p}_6) = \mathbf{v}$. Assume that at least one $\beta_i < 0$, since otherwise we have found the desired $\alpha_i' = \beta_i$. Put $\gamma = \max_{\beta_i < 0}\left(\frac{\alpha_i}{\beta_i}\right) < 0$, and assume $\gamma = \alpha_1/\beta_1$. Let $\alpha_i' = \alpha_i - \gamma\beta_i$ for $1 \leq i \leq 6$. Then

$$\alpha_1' \Gamma(\mathbf{p}_1) + \cdots + \alpha_6' \Gamma(\mathbf{p}_6) + \alpha_7 \Gamma(\mathbf{p}_7) + \cdots + \alpha_N \Gamma(\mathbf{p}_N) = (\alpha - \gamma)\mathbf{v}$$

Since $\alpha_1' = 0$, $\alpha_2', \ldots, \alpha_6' \geq 0$ and $(\alpha - \gamma) > 0$ we have reduced the number of vectors by at least one. Note that this procedure is reminiscent of the pivoting done in the Simplex method in that we reach lower dimensional facets of the bounding polyhedron at each step.

The other case of $\Gamma(\mathbf{p}_1), \ldots, \Gamma(\mathbf{p}_6)$ linearly dependent is similar. Therefore in $O(n)$ time, a twelve fingered closure grasp can be found.

Friction and other contact types

The number of fingers necessary for a closure grasp can be reduced in the presence of friction. Usually the Coulomb friction model for the surface contacts is assumed. A friction cone is associated with each contact, and the line of action of the force transmitted through the contact must lie within this cone.

In the plane, only three fingers are required for form closure and in $\mathbb{R}^3$, form closure is possible with four fingers, and these numbers are also lower bounds.

Let $\rho > 0$ be the friction coefficient. Form closure with n fingers under friction occurs when any arbitrary external wrench can be expressed as

$$\mathbf{w} = \sum_{i=1}^{n} f_i \left[\mathbf{w}_i + (\mathbf{n}'_i, \mathbf{n}'_i \times \mathbf{p}_i) \right], \tag{3}$$

where $f_i \geq 0$ and $\mathbf{n}'_i$ is normal to $\mathbf{n}(\mathbf{p}_i)$ with $|\mathbf{n}'_i| \leq \rho$, $(1 \leq i \leq n)$.

Markenscoff, Ni and Papadimitriou show the upper bound as follows. They first establish [MNP90, lemma 9] that a given gwasp with positively independent finger wrenches is a form closure under friction coefficient $\rho > 0$ if and only if it is a form closure with unbounded friction coefficient.

To see the if part, let 3 hold for any arbitrary wrench $\mathbf{w}$. By positive independence, there are positive numbers g_i, $1 \leq i \leq n$, such that $\mathbf{W} = \sum_1^n g_i \mathbf{w}_i = \mathbf{0}$. Let $h = \max_i(f_i|\mathbf{n}'_i|/(g_i\rho))$ and $f'_i = f_i + g_i h$. Then, adding h multiples of $\mathbf{W}$ to (3) we get,

$$\mathbf{w} = \sum_{i=1}^{n} f'_i \left[\mathbf{w}_i + \frac{f_i}{f'_i} (\mathbf{n}'_i, \mathbf{n}'_i \times \mathbf{p}_i) \right].$$

So $\mathbf{n}'_i$ can be replaced by $\mathbf{n}''_i = \frac{\mathbf{n}'_i}{1+g_i h}$ with $|\mathbf{n}''_i| \leq \rho$ as required. This process can however substantially increase the internal forces g_i.

The proof here is similar in spirit to the one given in [MNP90] and proceeds by again finding the largest sphere inscribed in the body. In general, this sphere touches the body at four points (three in the plane) at which the normals are positively spanning. The fingers are placed at those points and force closure is attained. Torque closure is obtained using only tangential forces due to friction. Assume the center of the sphere to be the origin. Let $\mathbf{n}_i$, $1 \leq i \leq 4$, be the normals at the contact points $\mathbf{p}_i$, with $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ linearly independent. Then put $\mathbf{n}'_i = \mathbf{n}_i \times \mathbf{n}_{i \bmod 3+1}$, $1 \leq i \leq 3$. The unit wrenches corresponding to forces along $\mathbf{n}_i$ are $\mathbf{w}_i = (\mathbf{0}, -\mathbf{n}_i \times \mathbf{n}'_i)$ since $\mathbf{p}_i = -|\mathbf{p}_i|\mathbf{n}_i$. An elementary calculation shows that $\mathbf{n}_i \times \mathbf{n}'_i$, $1 \leq i \leq 3$ are linearly independent. Hence $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ generates the space of pure moments.

The other cases of fewer contact points between the sphere and the body are treated similarly, and form closure with friction is achieved. Hence

> Any three dimensional object can be held with a closure grasp under friction by a positive grip of four fingers.

There are other contact types that should be included in the study of form closure grasps. Contacts such as line contacts, planar contacts, both frictional and non frictional, are described by Salisbury [MJKS85]. There are also different models of friction for materials such as rubbers, which usually display a coefficient of friction that varies as a function of the normal force [CAHK87]. Algorithms need to be developed for finding good grasps using those more general contact types.

4 Efficiency of a Closure Grasp

The techniques of the previous sections yield a method to synthesize at least *one* closure grasp of an object. However, in the absence of any measure of "goodness" for closures grasps, the synthesized grasp may not be very robust, thus, not useful in practice. This has motivated further research that attempts to quantify the goodness of closure grasps and to synthesize provably good closure grasps [KMY90].

One criterion for goodness is the "efficiency" of a grasp, which roughly measures the amount of external force and torque that can be resisted by applying at most a unit of force at each grasp point. Kirkpatrick, Mishra and Yap [KMY90] have shown that this efficiency metric can be derived from a stronger *quantitative version* of the Steinitz's theorem. Before we explore this connection, we need to introduce few notations, as follows.

For any set $X \subseteq \mathbb{R}$, let the *residual ball* of X refer to the maximal ball $B(X)$ centered at the origin $\mathbf{0}$ such that $B(X)$ is fully contained inside the convex hull of X. The *residual radius* of X, denoted $r(X)$ is the radius of this residual ball $B(X)$. Let

$$\begin{aligned} r_d(m, X) &= \max\{r(Y) : Y \subseteq X \text{ and } |Y| \leq m\}, \\ r_d(m) &= \min\{r_d(m, X) : X \subseteq \mathbb{R} \text{ and } \smallsetminus(\mathbb{X}) \geq \mathbb{1}\}. \end{aligned}$$

In the notation, we shall omit the subscript d, if the dimension is clear from the context. (In this paper, the interesting case is $d = 6$.) Thus the original Steinitz's theorem can now be interpreted to say that

$$r_d(2d) > 0.$$

A quantitative version of Steinitz's theorem provides more precise bounds for the number $r_d(m)$, when $m \geq 2d$.

Now, the achievable efficiency of a closure grasp with a m-fingered positive grip for a body B can seen to be the value, $r_6(m, \Gamma(\partial B^*))$. To see this, note that if we choose m points in $\Gamma(\partial B^*)$ with residual radius r then any wrench vector $\mathbf{v}$ whose Euclidean norm is at most r can be written as a convex combination of the m chosen points. So if $\mathbf{v}$ is any external wrench that is applied to the body B, and $\mathbf{v}$ lies in the residual ball of radius r, we can resist this external wrench by applying suitable forces (of magnitude at most 1) at the grasp points such that these forces sum to $-\mathbf{v}$; hence, we maintain the body in equilibrium. Thus, we see that the quantity $r_6(m)$ gives a universal measure for the efficiency of a closure grasp with $m \geq 12$ fingers. The special case, where $m = 2d$ had been studied by Bárány, Katchalski and Pach [BKP82]; they showed that

$$r_d(2d) > \frac{c}{(2ed)^{\lfloor d/2 \rfloor} d^2}.$$

These results seem to indicate that a twelve-finger positive grip, while sufficient to provide a closure grasp, may not be adequate to achieve a desirable efficiency. Kirkpatrick, Mishra and Yap [KMY90] have provided the following general bounds:

> *For any set $X \subseteq \mathbb{R}$ whose convex hull contains the unit ball B^d centered at the origin $\mathbf{0}$, we can find a set $Y \subseteq X$ of at most m points with residual radius $r(Y)$ of at least*
>
> $$1 - 3d\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}},$$
>
> *where m is assumed to be sufficiently large.*
>
> *Furthermore, let $X \subseteq \mathbb{R}$ be the set of all points on the surface of the d-dimensional unit ball centered at the origin $\mathbf{0}$. Then, every subset $Y \subseteq X$ of at most m points has a residual radius $r(Y)$ of at most*
>
> $$1 - \frac{1}{17}\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}}.$$

In summary, for sufficiently large m ($m \geq 13^d d^{(d+3)/2}$)

$$1 - 3d\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}} \leq r_d(m) \leq 1 - \frac{1}{17}\left(\frac{2d^2}{m}\right)^{\frac{2}{d-1}}.$$

This indicates an interesting *trade-off* between efficiency of a closure grasp and the number of fingers.

Another negative result has been shown by Meyer [Mey90] who shows that a hand with seven fingers, and a limit of 1 on the magnitude of forces it can apply, is weak in the following sense.

Let a hostile finger push on the body B at point $\mathbf{p}_h$. Let $\lambda(\mathbf{p}_h)$ be the maximum magnitude of the hostile finger force which can be opposed by the grasp. Then the strength of the grasp is $s = min_{\mathbf{p}_h \in \partial B}[\lambda(\mathbf{p}_h)]$.

Meyer proposes a family of rectangular parallelipipeds with one side of unit length and two smaller sides of length ϵ. He shows the strength is bounded by $5\sqrt{2}\epsilon$ for any given grasp, i.e. arbitrarily small.

The basic idea is that there must be two parallel faces with only one finger on each face, then the hostile finger can push in a direction perpendicular to those faces and produce sufficient torque.

Meyer's efficiency measure is similar to the preceding one, but takes into account the geometry of the object. However wrenches such as those due to acceleration are not considered.

Thus, it is dubious whether closure grasps can be employed in practice to implement control algorithms for dextrous manipulation, where no regrasping is necessary during the entire task execution.

However, if one allows large number of fingers, or allows frictional and/or soft contact models, there is a possibility for synthesizing moderately efficient closure grasps. Kirkpatrick et. al. have provided certain approximation algorithms for these problems and some related computational geometric problems. However, still much research is needed to provide practical algorithms.

Optimization

Ideally, one would like to produce grips that use both small forces and require low friction coefficients. These grips should of course be stable and withstand outside forces that depend on the manipulation task. Some of these conflicting goals have been considered, mostly in two dimensions.

In the case of three fingered grasps of planar polygonal objects, Markenskov and Papadimitriou [MP89] find a grasp that minimizes the forces required to balance a force along the third dimension applied to the center of gravity. This is done using frictional forces along the third dimension. Frictional forces in the plane are not considered. The forces are applied perpendicularly to the sides of the polygon and their lines of action meet at a point. The function minimized is a non-decreasing function of the forces, and in the presence of reflex vertices, the angles between their lines of action are minimized. This is done numerically.

Their procedure must however be repeated $O(n)$ times, where n is the number of sides of the polygon, to find the overall optimum.

They also define and optimize the following simplified version of form closure. A set of fingers achieves form *semiclosure* of a polygon if any wrench through the center of gravity can be balanced by positive forces normal to the perimeter. To find such a set of fingers that minimizes the forces required to balance a unit wrench, they formulate an optimization problem which is then solved using linear programming and binary search. This optimization procedure must again be applied for all quadruples of sides.

Ji and Roth [JR88] consider three fingered grasps in three dimensions, but have a reduction of the problem to two dimensions for most cases. The plane considered is the one containing the three finger contact points. They assume that the fingers are given, but their angle can vary, subject to the restrictions imposed by the friction cones. These angles are chosen so that the dependence of the internal forces on friction is minimized. It is shown using a geometric argument that this occurs when the angles between the lines of action and the normals at the corresponding points are equal. In addition, the relative magnitude of the forces is computed using the sine law applied to the force triangle. These can then be varied as needed to help resist disturbances and form a secure grasp.

Another approach, that of minimizing the friction coefficient μ needed to obtain an equilibrium grasp of a polygon, was taken by Schwartz and Sharir [SS87]. They consider the case of n fixed finger contacts, and compute the smallest friction coefficient μ^* that allows a such a grasp. The force $\mathbf{f}_i$ applied by finger i at point $\mathbf{p}_i$ must belong to the cone spanned by two vectors $\mathbf{v}_i^+(\mu) = \mathbf{n}_i + \mu\mathbf{t}_i$ and $\mathbf{v}_i^-(\mu) = \mathbf{n}_i - \mu\mathbf{t}_i$ for all i, where $\mathbf{t}_i$ is the unit tangent at $\mathbf{p}_i$. The condition to be satisfied is

$$\sum_{i=1}^{n} \alpha_i^+ \mathbf{u}_i^+(\mu) + \sum_{i=1}^{n} \alpha_i^- \mathbf{u}_i^-(\mu) = \mathbf{0}$$

where $\mathbf{u}_i^+(\mu)$ (resp. $\mathbf{u}_i^-(\mu)$) is the three dimensional vector $(\mathbf{v}_i^+, (\mathbf{v}_i^+ \times \mathbf{p}_i)_3)$ (resp. $(\mathbf{v}_i^-, (\mathbf{v}_i^- \times \mathbf{p}_i)_3)$) and $\alpha_i^+,\ \alpha_i^- \geq 0$ for all i. Since the α_i's can be normalized, we wish to express the origin as a convex combination of the vectors above. Given an optimal friction coefficient μ^*, the force targets would then be $\mathbf{f}_i = \sum_1^n \alpha_i^+ \mathbf{u}_i^+(\mu^*) + \sum_1^n \alpha_i^- \mathbf{u}_i^-(\mu^*)$.

Let $C(\mu) = \mathrm{conv}\ (\mathbf{p}_i : 1 \leq i \leq n)$. This set has the property of growing as μ is increased. If the exists a frictional grip with the contact points specified, the for μ sufficiently large, $\mathbf{0} \in C(\mu)$. The optimal value of μ occurs at the point where the boundary of $C(\mu)$ reaches the origin. Whether $\mathbf{0}$ lies in $C(\mu)$ can be checked by solving a linear program. This can be done in $O(n)$ time [Meg84]. A sophisticated searching procedure [Meg83] is used to find the optimum. These techniques exhibit very large constants in their running time and are not usable in practice, and simpler algorithms would need to be used.

They extend their technique to allow the points to be placed anywhere on selected edges of the polygon.

The results described above use linear programming, geometry and optimization techniques to obtain good grasps under various restrictions and various definitions of goodness. It would be interesting to extend those techniques to three dimensions and be able to optimize an efficiency measure that achieves a balance of the goals, balance that depends on the manipulation task at hand.

5 Finite Immobility

In the context of geometric immobility, issues such as efficiency do not arise as naturally. However, it remains open whether it is possible to determine the optimal number of points that can immobilize a given polygon in less than the trivial $O(n^3)$ time remains open [O'R90a], as does the analogous problem in d dimensions.

The largest inscribed sphere used in [CSU90] provides immobilizing sets but all such sets are not necessarily obtained in this manner. A first step in this direction has recently been made in [Tei91]. For polygons of n vertices, an $O(n^2 \log n)$ time algorithm is given to find the optimal number of immobilizing points that are not vertices.

In fact even testing for immobility in the geometric model is more difficult than in the physics based model. A simple construction establishes a connection between testing for finite immobility and the problem of testing whether a graph, realized by rigid rods connected at the vertices is rigid. This problem is described by Roth in [Rot81]. The same construction seems to establish a similar connection between force/torque closure and infinitesimal graph rigidity. The latter can be easily tested by computing the rank of a certain matrix derived from the graph. But again, "rigidity" is harder to test for than "infinitesimal rigidity".

7 Conclusion and Future Research

So far, we have only discussed the static problems (i.e. grasping) in dexterous manipulation. There are still a large number of exciting and challenging questions related to the dynamic aspects of dexterous manipulation. Examples of such questions include fine-manipulation of objects by continuous finger-gaiting [HLMT90], dexterous adjustment of grasp configuration [LCS89], prehension strategy and motion planning [SHS86, SY87] and controlled motion of grasped object [LHS89].

A variety of issues involving high-level task planning remain largely unexplored. A thorough understanding of these questions are crucial to bring the science of dexterous manipulation into practice.

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