Some Constructions in Rings of Differential Polynomials*

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August 28, 1995

Introduction

In the recent years, motivated by the success of various algorithms and techniques in computational algebraic ring theory, the researchers in the Computer Algebra community have turned their attention to the task of extending these ideas to differential ring. Appropriate analogs and generalizations of standard bases, Buchberger's completion algorithm, etc. are however yet to be found. (See for example [1], [2], [16] and [19]).

The reasons for this interest are both practical and theoretical. There is, in fact, a growing effort to use differential algebra in order to solve problems in Control theory, Dynamical systems and Robotics. From the theoretical point of view, it is equally important that we understand the precise relation between 'old' constructive methods (Ritt-Seidenberg algorithm [5, 24]) and the recent Gröbner bases-like approach.

A constructive study of differential algebra may also give new insight into its quite complicated structures. For example, rings of differential polynomials are not Noetherian, hence differential ideals can be much more complex than algebraic ideals. An example, reported later on, shows that this difference implies that there are differential ideals that are not recursive!

On the other hand the structure of differential ideals is not completely unruly, and one can hope to characterize classes of differential rings and of ideals for which suitable algorithmic techniques can be developed. The concept of H-bases for differential ideals is introduced in the second part of this paper and it constitutes a contribution to this direction of research.

The differential algebras considered here are commutative rings of differential polynomials in several differential indeterminates over a field of constants, a

*This research was supported in part by NSF Grants Numbers: CCR-9002819, IRI-9003986, ONR Grant Number: N00014-89-J3042 and an Italian CNR Fellowship.

particular case of the algebras introduced by the classical works of Janet [8], Kolchin [11], Riquier [20] and Ritt [22]. These objects should not be confused with the rings of the differential operators whose constructive aspects have been studied, for example, in [3] and [4].

The rest of this paper is organized as follows: The first section fixes the standard notations and recalls some classical notions in Differential Algebra. The second section presents an example of a nonrecursive differential ideal in the ring $\mathbb{Z}\{x\}$. This example, which generalizes the one due to Ritt and Kolchin, shows that the problem of deciding the membership in a differential ideal is not in general algorithmically solvable. It however leaves open the question whether the membership question for a recursively generated differential ideal can be solved by a recursive procedure. The third section introduces the concept of H-bases for differential ideal, in analogy with Macaulay's original definition of H-bases for algebraic ideals [13] (see also [15] and [23]).

These bases can be regarded as a special kind of standard bases (cfr. [1], [16] and [17]). Using such bases, one can introduce a particularly simple procedure to test the membership of a differential polynomial in a differential ideal. Unfortunately, for general differential ideals, it is not known if there is a finite method capable of computing an H-basis, or even a useful subset of it.

In the special case of ideals with an isobaric basis, however, H-bases can be computed and we present an effective procedure to decide the membership in such ideals. It is still unresolved if a finitely generated (respectively, recursively generated) differential ideal has a finite (respectively, recursive) H-basis.

1 Preliminaries

In order to keep the following expositions reasonably self-contained, we shall introduce in this section most of the standard notations to be used later. For those notations and definitions that are not explicitly mentioned here, refer to [9], [11] or [22].

All our rings are assumed to be commutative and with unity.

Definition 1.1 A ring R is said to be a *differential ring* if there exists a differential operator from R to R, i.e., a map $d: R \to R$ such that, for all α and β in R:

- d is *linear*, i.e. $d(\alpha+\beta) = d(\alpha) + d(\beta)$;
- $d(\alpha\beta) = d(\alpha)\beta + \alpha d(\beta)$.

Example 1.1 Any algebraic ring R, with d defined as the trivial derivation, i.e. $d(r) = 0$ for all $r \in R$, is a differential ring.

Example 1.2 The ring of analytic functions over a domain of $\mathbb{C}$ is a differential ring.

Definition 1.2 A subset I of a differential ring R is a *differential ideal* if it is an algebraic ideal of R and moreover, it is closed under the d operator, i.e. if $d(I) \subseteq I$.

If S is a subset of R and I is the minimal differential ideal of R containing S, then S is said to be a system of generators for I, or equivalently I is said to be the ideal generated by S. If $S = \{f_1, \ldots, f_n\}$ then I is denoted by $[f_1, \ldots, f_n]$.

Since R is a particular algebraic ring one can also consider the algebraic ideal J generated by S. This ideal is sometimes denoted by $(f_1, \ldots, f_n)$. Note that, in general, $(f_1, \ldots, f_n) \neq [f_1, \ldots, f_n]$.

Definition 1.3 A differential ring R, with derivation d, is said to be *computable* if it is possible to solve algorithmically the following problems:

- decide if r is an element of R;
- compute for any element $r \in R$ the element $-r$;
- compute for any pair $(r, s) \in R \times R$ the element $r + s$;
- compute for any pair $(r, s) \in R \times R$ the element rs;
- decide if $r \in R$ is equal to the zero element of R;
- compute $d(r)$ for any element $r \in R$;

Differential rings of particular interest are the ones constructed from a differential ring R by adjoining some differential indeterminates, as follows.

Definition 1.4 Let R be a differential ring. Consider the ring of polynomials $A = R[x_0,\, x_1,\, \ldots,\, x_n,\, \ldots]$ with a denumerable number of variables. A is a differential ring once the derivation d' on R is extended to a derivation d on A:

- $d(r) = d'(r)$ for all the elements $r \in R$;
- $d(x_i) = x_{i+1}$ for $i \geq 0$.

After renaming x_j as $d^{(j)}x$ (the derivation of order 0 is assumed to be the identity map), A can be denoted by $R[x,\, dx,\, d^{(2)}x,\, \ldots,\, d^{(n)}x,\, \ldots]$ or by $R\{x\}$; it is called the *ring of differential polynomials* in x over R.

It is possible to iterate the definition above to adjoin more variables to R, obtaining the differential ring $R\{x_1, \ldots, x_n\}$ of the differential polynomials in $x_1, \ldots, x_n$ over R.

Remark 1.1 It can be immediately verified that if R is a computable differential ring, then so is $R\{x_1, \ldots, x_n\}$.

Remark 1.2 A differential polynomial can be written uniquely as a sum of terms. A term is the product of an element of the ring (the coefficient) and a monomial of the indeterminates and their derivatives.

Definition 1.5 Let M denote the set of all monomials in the differential ring $R\{x_1, \ldots, x_n\}$. Consider the map $w: M \to \mathbb{R}$ so defined that:

- $w(x_i) = m_i$ with $m_i > 0$ for $i = 1, \ldots, n$;
- $w(d^{(k)}x_i) = (k + m_i)$ for any integer $k > 0$ and for $i = 1, \ldots, n$;
- For any monomial $m \in M$, $w(m) = \sum_i w(f_i)$ where f_i's range over the factors of m containing a single indeterminate or a derivative.

The function w is called a *weight* of the monomials of the differential ring $R\{x_1, \ldots, x_n\}$. The weight of a differential polynomial is the maximum weight of its power products.

A differential polynomial whose monomials have all the same weight, with respect to some weight function w is called *isobaric*. Any polynomial can be uniquely written as the sum of isobaric polynomials that are called its isobaric components. The isobaric component of maximum weight is called *leading isobaric component* (or *head*) of f; it is denoted as $h(f)$. Notice that the isobaric component of a polynomial are not, generally, monomials, but isobaric polynomials.

A differential ideal I is called isobaric if, whenever a differential polynomial f is in I, all of its isobaric components belong to I.

Remark 1.3 It is easy to verify that any weight function defined as above has the following properties:

- $w(fg) = w(f) + w(g)$, for all $f, g \in R\{x_1, \ldots, x_n\}$;
- $w(d(f)) = w(f) + 1$, for all $f \in R\{x_1, \ldots, x_n\}$;
- The set S_k of differential power products of weight k is finite, for any k.

In a fashion similar to the case of homogeneous ideals of polynomial rings, it can be shown that:

Proposition 1.1 *A differential ideal I in a ring of differential polynomials with constant coefficients is isobaric if and only if it has a system of isobaric generators.* ∎

Note that the above proposition, in general, is not true if the derivatives of the coefficients of the polynomials are not zero. In fact in this case the derivative of an isobaric polynomial may not be isobaric, as in the following example.

Example 1.3 Consider the ring $R = \mathbb{Q}\langle e\rangle$, where $d(e) = e$. Then the derivative of the differential isobaric polynomial $x + ey$ in the ring $R\{x, y\}$ is the polynomial $dx + edy + ey$ whose isobaric components are $dx + edy$ and ey. But ey is not in the ideal $I = [x + ey]$. I has an isobaric basis, but is not isobaric.

2 Nonrecursive Differential Ideals

The theory of rings of differential polynomials can be seen as a generalization of the theory of polynomial rings. It is, hence, quite natural to try to prove for such rings results similar to the ones known for polynomial rings. Unfortunately, many fundamental properties of algebraic rings do not hold any more when generalized to a differential context, or their validity can be assured only under the assumption of some restrictive hypotheses.

This section reports one of such differences and exploits it to prove the existence of non-recursive differential ideals of the ring $R\{x\}$ even when R is a computable differential ring.

In the constructive theory of polynomial rings, a central role is occupied by the Hilbert basis theorem which, assures the existence of a finite system of generators for an ideal in $R[x_1, \ldots, x_n]$, provided that the ring R is Noetherian. This theorem is useful to prove the finiteness of several algebraic constructions. Unfortunately for rings of differential polynomials only a weaker version of the Hilbert basis theorem can be proved: *The Ritt-Raudenbusch Basis Theorem.*

Theorem 2.1 *If R is a differential ring such that every strictly ascending chain of* radical *ideals of R is finite, then so does the differential ring $R\{x\}$.* ■

For a proof of this theorem see [22] and [9]. The Ritt-Raudenbusch theorem cannot be improved as the following example shows (see [21] p.12–13, [10]).

Example 2.1 Consider the ring of the integers $\mathbb{Z}$. It can be considered as a differential ring with the trivial derivation d, i.e., the derivation satisfying: $d(m) = 0$ for any $m \in \mathbb{Z}$ (see Example 2.1). Let $\mathbb{Z}\{x\}$ be the ring of differential polynomials in one indeterminate x over $\mathbb{Z}$.

Assume that the symbol $d^{(0)}$ denotes the identity map over $\mathbb{Z}\{x\}$. Define the differential polynomials f_i as follows:

$$f_i = \left(d^{(i)}x\right)^2 \qquad\qquad i \geq 0.$$

Consider the countable family of non-radical, isobaric differential ideals defined in the following way:

$$\begin{aligned} I_0 &= (f_0) \\ I_1 &= (f_0, f_1) \\ &\vdots \\ I_n &= (f_0, f_1, \ldots, f_n) \\ &\vdots \end{aligned}$$

Claim. The ideals I_j's form a strictly ascending infinite chain:

$$I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_n \subsetneq \cdots$$

It is immediate to verify that the inclusion relations above hold, hence to prove the claim one needs to show only that all such inclusions are proper, i.e., that for each index $i > 0$ there is a differential polynomial in I_i which is not contained in I_{i-1}. The following remarks show that the differential polynomial $f_i \in I_i$ is not contained in the ideal I_{i-1}.

Remark 2.1 The polynomials f_i's with $i = 0, 1, \ldots,$ and their derivatives are homogeneous and of degree 2. Moreover, if the weight assigned to the variable x is 1 the f_i's are isobaric of weight $2i + 2$. Thus, Their k^{th} derivative will be isobaric of weight $2i + 2 + k$.

Remark 2.2 Suppose that $f_n = (d^{(n)}x)^2$ is in I_{n-1}, i.e., there exist differential polynomials $\alpha_{i,j}$'s such that the following expression holds:

$$f_n = \sum_{i=0}^{n-1} \sum_{j=0}^{k_i} \alpha_{i,j} d^{(j)} \left[\left(d^{(i)} x \right)^2 \right]. \tag{1}$$

If the $\alpha_{i,j}$'s have positive degree, their products with the f_i's and their derivatives will contain monomials of degree higher than two. Since f_n has degree two, these monomials must cancel. The monomials of positive degree in the $\alpha_{i,j}$ are, thus, redundant, i.e., the $\alpha_{i,j}$ can be supposed to be in $\mathbb{Z}$.

Remark 2.3 Since f_n is isobaric, it is immediate to see, from the previous remark, that at most one occurrence of $d^{(k_i)} f_i$, $i = 0, 1, \ldots, n-1$, can appear in (1). Moreover, k_i must be equal to $2n - 2i$. Hence if f_n is in I_{n-1} the following equation must hold:

$$f_n = c_0 d^{(2n)} f_0 + c_1 d^{(2n-2)} f_1 + \cdots + c_{n-1} d^{(2)} f_{n-1}, \tag{2}$$

where c_i are integers.

Remark 2.4 The polynomial $d^{(2n)} f_0$ contains the monomial $x d^{(2n)} x$. Since this monomial does not appear in any other addenda of the expression (2) and does not appear in f_n it follows that c_0 must be zero.

An analogous reasoning proves that $c_1, c_2, \ldots, c_{n-1}$ must all be zero. This is a contradiction, because f_n is not zero, and the claim is proved.

Following the same argument as above, one can prove a slightly stronger proposition:

Proposition 2.1 *Let S be a subset of $\mathbb{N} \cup \{0\}$ and I_S, the differential ideal generated by the set $\{f_i : i \in S\}$. Then*

$$f_j \in I_S \quad \Leftrightarrow \quad j \in S. \quad \blacksquare$$

The proposition above leads to a family of examples of nonrecursive ideals in $\mathbb{Z}\{x\}$.

Theorem 2.2 *Let S be a nonrecursive subset of $\mathbb{N} \cup \{0\}$. Then there is no algorithm to decide if a given differential polynomial g is in the ideal I_S.*
Proof. Assume to the contrary, i.e., there is an algorithm that decides if a given polynomial g is in I_S. This is impossible, as it would lead to an algorithm that decides if a given integer j is in the nonrecursive set S. (It suffices to test if the differential polynomial f_j is in I_S.) $\blacksquare$

The previous theorem rules out the existence of an algorithm capable of deciding the membership problem for general differential ideals. Thus, it is natural to investigate if there is such a decision algorithm for some restricted class of differential ideals (for instance, a differential ideal with a recursive [or even finite] set of generators) .

3 H-bases of Differential Ideals

In 1929, Macaulay in his book *Algebraic theory of modular systems* [13] introduced the concept of H-bases (the 'H' stands in honor of Hilbert) for ideals in the ring of polynomials. He also gave, in the same place, a sketch of an algorithm to compute an H-basis of an algebraic ideal starting from a finite set of its generators. Macaulay's construction has received new attention in the recent developments of computational algebra. It has been investigated in relation with the Gröbner bases construction and has been applied to general graded structures ([15], [23]).

In this section, we propose a generalization of the H-bases to differential ideals in rings of differential polynomials with coefficients in a computable differential field of characteristic zero.

The concept of H-bases proposed in this paper is not, generally, an effective one. It is not known, as a matter of fact, if finitely (respectively recursively) generated differential ideals have a finite (respectively recursive) H-basis. Moreover no effective method is known for the computation of the finite set of all the differential polynomials with bounded weight in an H-basis starting from a finite (recursive) set of generators of a general differential ideal.

Consider the ring $A = K\{x_1, \ldots, x_n\}$ together with a weight function w defined on it. A is isomorphic to the direct sum $\oplus_{i=0}^{\infty} A_i$, where A_i is, for any integer i, the finite dimensional K-vector space of the isobaric polynomials of weight i.

Definition 3.1 Let I be a differential ideal in $R\{x_1, \ldots, x_n\}$. The ideal $H(I)$, is the *algebraic* ideal generated by the set of all the leading isobaric components of the differential polynomials in I.

By definition a differential polynomial f is in $H(I)$ iff all of its isobaric components are in $H(I)$.

Notice that $H(I)$ is not in general a differential ideal, unless the field K is a field of constants.

Let $H(I)_i = H(I) \cap A_i$, i.e., $H(I)_i$ is the subspace of A_i spanned by the isobaric differential polynomials of $H(I)$ of weight i.

The Hilbert-like function $\phi_{I,w}(i)$ is defined to be the dimension of $H(I)_i$ as a vector space over K.

Remark 3.1 The function $\phi_{I,w}$ defined above is analogous, for differential ideals, to the Hilbert function for algebraic ideals. It is, however, not a good idea to call it by the same name; the designation "Hilbert function," in Differential Algebra, is usually reserved for an unrelated concept.

It is not known if $\phi_{I,w}$ has the same regularity properties as its algebraic analog. The existence of something similar to a "*Severi's regularity bound*" for such functions would imply that an H-basis is a recursive set, generalizing the similar results obtained for Gröbner bases. Unfortunately, such functions seem to be extremely complicated, and have been investigated, because of their combinatorial interest, only for few special classes of differential ideals (see [12], [14] and [6]).

Remark 3.2 Given a basis S for the differential ideal I, consider the *algebraic* ideal $H(S)$ generated by the leading isobaric components of the polynomials in the set $\bar{S} = \{g : g = d^{(k)}s \text{ with } s \in S, \ \ k \in \mathbb{N} \cup \{0\}\}$.

Generally, $H(S)$ is properly contained in $H(I)$. Consider for example the ideal $I = [x]$ in the ring $\mathbb{Z}\{x\}$ and the basis $S = \{x + dx, x - dx\}$.

Notice again that, as above, $H(S)$ is not in general a differential ideal, unless the field K is a field of constants.

Imitating the algebraic case, it is then natural to introduce the following definition.

Definition 3.2 A basis S of the differential ideal I is said to be an *H-basis* if

$$H(S) = H(I).$$

It is clear that the property of being an H-basis depends on the weight function considered over the ring.

Remark 3.3 The definition of H-bases given above is, in some sense, *purely algebraic.* Consider an algebraic commutative ring A, a ring of algebraic polynomials with a denumerable number of variables $x_1, x_2, \ldots, x_i, \ldots$.

The weight function defines a graded structure on A and one can introduce in the obvious way the concept of an algebraic H-basis for the ideals in the ring A: A basis $\bar{S}$ is an H-basis for the ideal I if the initial forms (i.e., the leading isobaric components) of the elements in $\bar{S}$ generate the algebraic ideal of the initial forms of the polynomials in I.

Now in the special case, where the variable x_i is assumed to mean $d^{(i)}x$, the straightforward restriction to differential ideals (which are particular algebraic ideals of A) results immediately in the definition 3.2.

Remark 3.4 The point of view taken in remark 3.3 is somewhat unsatisfactory. In fact, one may wish to exploit the differential structure to give a more 'compact' description of an H-basis. But, this is complicated by the fact that, for differential ideals, the set $\bar{S}$ is infinite.

Restating the definition of H-bases given above in terms of differential structures, however, leads to certain peculiarities, distinguishing the case of *constant* coefficients from that of *nonconstant* coefficients.

If K is a *field of constants*, then the operations of extracting the leading isobaric component of a differential polynomial, and taking its derivative do commute. Thus, in this case, the algebraic ideals generated by the leading isobaric components of the polynomials in I and in $\bar{S}$ are *differential ideals*. Moreover, $H(S)$ is generated, as a differential ideal by the leading isobaric components of the elements of S. The definition 3.2 can be restated just in terms of differential ideals, if, indeed, the underlying field is a field of constants.

On the other hand, if K is *not a field of constants*, derivation and extraction of the leading isobaric components do not, in general, commute. Then the algebraic ideals $H(I)$ and $H(S)$ may not be differential ideals.

To get a 'differential' description of H-bases, in this case, one may adopt the point of view of [16] and [17]. Namely, one introduces a new derivation d_* on the ring A such that $d_*(K) = \{0\}$. Otherwise d_* is equal to d.

Define $H_*(I)$ to be the differential ideal (with respect to this *new* differential structure) generated by the leading isobaric component of the elements of I (which, under this new derivation, is just an algebraic ideal), and define $H_*(S)$ to be the differential ideal generated by the leading components of the elements in S.

It is easy to verify that the condition for S to be an H-basis in definition 3.2 is equivalent to the relation $H * (S) = H_*(I)$, which is an equality between differential ideals.

Remark 3.5 A very immediate and naive "procedure" to compute an H-basis starting from a finite set S of generators of I is the following: Determine for any weight k the set $H(S)_k$ and check for any linear relations (i.e., the syzygies) among the elements of such a set. Such relations will provide new generators (of smaller weight) which can be added to S in an effort to "complete" S to an H-basis.

The drawback of this method is that no bound is known for the weight of the syzygies to be computed in order to complete the set of all the polynomials of a given weight in an H-basis. The method proposed is, thus, not effective. However, it may sometimes provide useful informations about the ideal I.

Moreover, if I has a finite H-basis it will be eventually constructed by this procedure, but we do not know of any algorithmic way to check when this happens.

Let S be a set of differential polynomials such that for any fixed integer k there are only finitely many elements in S with weight k. It is possible to introduce a variant of "*rewriting procedure*" for differential polynomials using these elements of S.

Let f be an isobaric differential polynomial. Suppose that $f_1, \ldots, f_p$ are the elements of S, each of weight $w(f)$: Their leading isobaric components generate a subspace V of the vector space W of all the isobaric polynomials of weight $w(f)$. Because the field of the coefficients is computable it is possible to find $h_1(f)$ and $h_2(f)$ such that $h(f) = h_1(f) + h_2(f)$, with $h_1(f)$ in V and $h_2(f)$ in the orthogonal complement of V. In particular, it is possible to compute elements a_i in K such that

$$h_1(f) = a_1 h(f_1) + \cdots + a_p h(f_p).$$

The differential polynomial

$$f' = f - a_1 f_1 + \cdots + a_p f_p$$

is said to be a *reduct* of f modulo S. The "*reduction*" relation is denoted by the following notation:

$$f \rightarrow^S f'.$$

The reduction process can be generalized to an arbitrary differential polynomial g, by simply applying it to each one of the isobaric components of g.

The symbol "$\rightarrow^S_*$" denotes the transitive closure of the relation "$\rightarrow^S$." A simple, but tedious, argument shows that no polynomial f can lead to an infinite chain of reductions.

The reduction process gives a canonical simplifier for differential polynomials modulo a differential ideal I, when S is an H-basis of I. This can be proved using the well known arguments from the theory of standard bases: The proof uses only the *algebraic* properties of S (in the sense of remark 3.3).

Proposition 3.1 *Let I be a differential ideal; S, an H-basis for I, and f, a differential polynomial. Then f is in I if and only if any maximal chain of reductions with respect to S ends with 0.* ■

Remark 3.6 Notice that the only elements of an H-basis involved in the process of reducing a polynomial of weight k, are the elements of weight less than or equal to k. Hence, if for any k, an algorithm is known to compute the subset S_k of all the polynomials with weight less than or equal to k in an H-basis of I, then the membership in I can be effectively decided. In fact, this observation about H-basis applies *mutatis mutandis* to any system of differential rewriting-rules.

In the algebraic case, a homogeneous basis for an ideal I, is an H-basis; the statement generalizes to differential ideals with differential isobaric bases, as the following proposition shows.

Proposition 3.2 *Let I be a differential ideal in $K\{x_1, \ldots, x_n\}$. Let S be a set of isobaric polynomials which is also a basis of I. Then S is an H-basis of the ideal I.* ■

Combining the above proposition with our previous observations we get the following immediate corollary:

Corollary 3.1 *Let I be a differential ideal of the ring $K\{x_1, \ldots, x_n\}$, with a recursive set of generators S of isobaric differential polynomials. Further, assume that there is an algorithm to enumerate the set of all elements of any nonnegative integral weight n in S.Then*

- *I is a recursive subset of the ring $K\{x_1, \ldots, x_n\}$.*
- *The function $\phi(I, w)$ is computable.* ■

The corollary is not completely new (see for example [1],[16]) but the proof presented here is very elementary and the method proposed works in the general case of ideals with an infinite set of generators.

Remark 3.7 Since the isobaricity of differential polynomials depends on the particular weight function considered, one may ask if it is always possible to find a weight function and a basis of isobaric polynomials for a differential ideal I. The answer to this question is *no*. In the case of constant coefficients the existence of such basis, in fact, would imply that the ideal I is isobaric and there are examples of differential ideals in such rings that are not *"isobarizable."*

Acknowledgements

We wish to thank an anonymous referee for pointing out a subtle error in an earlier version of the paper and for suggesting ideas that led to a more general definition of H-bases.

References

[1] G. Carrá Ferro, *Gröbner Bases and Differential Ideals*, LNCS, Proceedings of AAECC5, 129-140, Springer-Verlag 1987.

[2] G. Carrá Ferro, *On Term-Orderings and Rankings*, preprint, 1990.

[3] F. Castro, *Théorème de division pour les opérateurs différentiels et calcul des multiplicités,* thèse de troisième cycle. Paris VII, Oct. 1984.

[4] A. Galligo, *Some Algorithmic Questions on Ideals of Differential Operators* LNCS, Proceedings EUROCAL 85, 413–421, Springer-Verlag 1985.

[5] G. Gallo and B. Mishra, *Efficient Algorithms and Bounds for Wu-Ritt Characteristic Sets*, in Proc. of MEGA 90, Castiglioncello, Italy, 1990, Birkhauser.

[6] R.M. Grassl, *Polynomials in Denumerable Indeterminates*, Pacif. Jour.of Math, 97 (1981) 2, 415–423.

[7] A.P. Hillman and R.M. Grassl, *Slicing Skew-tableau Frames*, Europ.Jour. of Combin. (1982) 3, 143–151.

[8] M. Janet, *Leçons sur le Systèmes d'équations aux Dérivées Partielles*, Gauthier-Villars, Paris, 1929.

[9] I. Kaplansky, *An Introduction to Differential Algebra*, Hermann 1957.

[10] E.R. Kolchin, *On the Basis Theorem for Differential Systems*, Trans. AMS, 52, 115–127, 1942.

[11] E.R. Kolchin, *Differential Algebra and Algebraic Groups*, Academic Press, New York 1973.

[12] H. Levi, *On the Structure of Differential Polynomials and on their Theory of Ideals*, Trans. AMS, 51, 326–365, 1942.

[13] F.S. Macaulay, *Algebraic Theory of Modular Systems*, Cambridge University Press, 1916.

[14] D.G. Mead, *Differential Ideals*, Proc. AMS 6 (1955), 420–432.

[15] H.M. Möller and F. Mora, *New Constructive Methods in Classical Ideal Theory*, Journal of Algebra 100 (1986).

[16] F. Ollivier, *Le problème de l'identifiabilité structurelle globale*, Doctoral Dissertation, Paris 1990.

[17] F. Ollivier, *Standard bases of Differential ideals*, in Proc. AAECC 8, Tokyo, Japan 1990, Springer Verlag.

[18] F. Ollivier, *Canonical bases: relation with standard bases, finiteness conditions and Application to tame automorphisms* in Proc. MEGA '90, Castinglioncello, Italy, 1990, Birkhauser.

[19] J.F. Pommaret, *Systems of Partial Differential Equations and Lie Pseudo-Groups*, Gordon and Breach, New York 1978.

[20] F. Riquier, *Les Systèmes d'équations aux dérivées partielles*, Gauthier-Villars, Paris 1910.

[21] J.F. Ritt, *Differential Equations from the Algebraic Standpoint* AMS Coloq. Publ. 14, New York 1932.

[22] J.F. Ritt, *Differential Algebra*, AMS Colloq. Publ. 33, New York 1950.

[23] L. Robbiano, *On the Theory of Graded Structures*, Journal of Symbolic Computation 2, 1986.

[24] A. Seidenberg, *An Elimination Theory for Differential Algebra*, Univ. of California at Berkeley, Publ. in Math. 3, 2 31–65, 1956.