On the Existence and Synthesis of Multifinger Positive Grips

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ABSTRACT *We study the criteria under which an object can be gripped by multi-fingered dexterous hand, assuming no static friction between the object and the fingers; such grips are called* positive grips. *We study three cases in detail:* (i) *the body is at equilibrium,* (ii) *the body is under some constant external force/torque, and* (iii) *the body is under a varying external force/torque. In each case we obtain tight bounds on the number of fingers needed to obtain grip.*

We also present efficient algorithms to synthesize such positive grips for bounded polyhedral/polygonal objects; the number of fingers employed in the grips synthesized by our algorithms match the above bounds. The algorithms run in time linear in the number of faces/sides.

The paper may be of independent interest for its presentation of algorithms arising in the study of positive linear spaces.

Work on this paper has been supported by Office of Naval Research Grant N00014-82-K-0381, National Science Foundation Grant No. NSF-DCR-83-20085, and by grants from the Digital Equipment Corporation, and the IBM Corporation.

1 Introduction

In the recent past, several researchers have proposed articulated, multifingered hands to augment the dexterity and sensitivity of a multi-linked robot arm. It has been suggested that such a hand will be capable of performing an extended class of manipulation tasks that have hitherto proven unwieldy for the state-of-the-art manipulators with simple grippers. Notable among such hands are Utah/MIT Dexterous Hand([10]), Stanford/JPL Hand ([19]), the Okada hand ([16]), the Asada hand ([1]).

In this paper, we envisage an idealized dexterous hand, consisting of several independently movable force-sensing fingers. These fingers move as points in the plane or in three-dimensional space. Here, we study the problem of *grip selection* for an object in the absence of static friction between the surface of the object and the fingers. Such non-frictional grips are called *positive grips*. Since the fingers are assumed to be point fingers, a finger can only apply a force on the object along the surface-normal at the point of contact, directed inward.

If the shape of the object is precisely known then the problem of *grip selection* reduces to that of choosing the following parameters:

1. Grip Points: The position vectors, $\mathbf{r}_1, \ldots, \mathbf{r}_l$; $\mathbf{r}_i$ corresponds to the position on the object K at which the i^{th} finger makes contact with the object.

2. Force Targets: The magnitudes of the forces, $\alpha_1, \ldots, \alpha_l$; α_i corresponds to the magnitude of the force applied by the i^{th} finger along the unit surface normal $\mathbf{n}(\mathbf{r}_i)$ of the object at $\mathbf{r}_i$, directed inward. Moreover, since the forces must be unisense, α_i must be non-negative.

These parameters must satisfy the following conditions of equilibrium:

$$\begin{aligned} F + \textstyle\sum_{i=1}^{l} \alpha_i\, \mathbf{n}(\mathbf{r}_i) &= \mathbf{0}, \\ \tau + \textstyle\sum_{i=1}^{l} \mathbf{r}_i \times \alpha_i\, \mathbf{n}(\mathbf{r}_i) &= \mathbf{0}, \end{aligned} \tag{1}$$

or,

$$[F, \tau] + \sum_{i=1}^{l} [\mathbf{n}(\mathbf{r}_i), \mathbf{r}_i \times \mathbf{n}(\mathbf{r}_i)] = \mathbf{0}, \tag{2}$$

where F and τ are respectively the external force and torque acting on the object. All the vectors, given above, are assumed to be expressed with respect to some arbitrary but fixed frame of reference.

Hence the natural questions that arise in this context are the following:

- *Can an arbitrary object be gripped positively with a finite number of fingers?*
- *If so, what are the grip points and the force targets for such a grip?*

We show that almost every smooth object allows a positive grip with only a bounded number of fingers, depending on whether the object is two- or three-dimensional, respectively. We also characterize exactly those 'problematic' surfaces that do not allow a positive grip irrespective of the number of fingers employed. In addition, we present a simple algorithm to compute the grip points and the force targets for a polyhedral object; the algorithm runs in time linear in the number of faces of the object.

Several other researchers have investigated various problems related to grasping, stable prehension, dexterous manipulation, etc.

Hanafusa and Asada [9] introduced the concept of a *stable prehension*—a prehension is called stable in the following case: when a relative position between the hand and the object deviates from a certain situation, a restoring force is generated by the fingers so that the relative position is brought back to the original situation. They presented a heuristic to determine the most suitable grasp positions for a hand consisting of three fingers and with three degrees of freedom. Liyangi and Kohli [12] also gave a similar heuristic for a constant-force design. Baker, Fortune and Grosse [2] pointed out that the Liyangi/Kohli approach may not find the most stable grip positions, and also presented a method which achieves a stable positive grip for two-dimensional polygonal objects with a hand consisting of three spring-loaded fingers and with five degrees of freedom. Their algorithm has a time-complexity of $O(n \log n)$, if the polygon has n sides. However, the grip positions they determine, may not be able to resist an external force or torque in the absence of static friction.

Kerr and Roth [11] investigated the problem of determining how hard to squeeze an object in order to ensure a secure grasp (the squeezing of the object is formally termed the *application of internal grasp forces*). Their model of hand is somewhat more realistic than that studied by some other authors, since it incorporates hands with soft fingers, thus allowing it to apply both forces and torques at the points of contact. Their work follows closely on Salisbury [17] and Salisbury and Craig [18].

In [14], Nguyen studied the synthesis of grasps for two-dimensional objects, which attain force-closure or torque-closure, that is, the grasp can resist arbitrary translation or rotation of the gripped object; the contacts used may be point or edge (with and without friction), or soft. In [15], he investigated the issues concerning stability of such force-closure grasps.

A general discussion of the problem of manipulation is found in Schwartz [20]. Schwartz and Sharir [21] investigated the issue of computing force targets when the gripping positions are already specified. Their result is of the most interest when the pre-specified points do not allow a positive grip; they present an $O(d \log d \log\log d)$-time algorithm ($d =$ number of fingers) to compute the smallest coefficient of static friction for which it is possible to assign a set of finger-force targets which hold the object at equilibrium and such that each individual force lies within the corresponding cones of friction.

In contrast to the bulk of the research in this area, this paper focuses on the problem of analyzing and synthesizing *at least one* positive grip. We operate within a general mathematical setting, which handles two- and three-dimensional cases uniformly. However, for the sake of simplicity, we restrict ourselves to the simplest kind of grip, i.e., a positive grip with point-contacts, and neglect issues such as, stability of the grasp, grasp planning etc.

The paper is organized as follows: In section 2, we study the criteria under which a body can be gripped positively and obtain tight bounds on the number of fingers required for each type of positive grips. We define and exactly characterize the objects that do not allow a force/torque closure grasp. Section 3 presents algorithms to synthesize positive grips for bounded polyhedral objects. We close the paper with some discussion of related open problems.

2 Existence of Positive Grips

2.1 Mathematical Preliminaries

We make the following simplifying assumptions:

Assumption 2.1

1. *Each finger can be positioned accurately, subject to only a small deviation, due to a lumped compliance between the finger 'top' and 'bottom.'*
2. *The object K to be gripped is a closed bounded connected domain in two- or three-dimensional space and has a piecewise smooth boundary Σ. Without loss of generality, we also assume that* Complement(K) *is connected, since any purely internal 'holes' in K can be filled in.*

3. *Each finger can apply only a single pure force, normal to surface through a frictionless point contact.*

Let K be an object with a piecewise smooth boundary Σ. Let $\mathbf{\Gamma}$ represent a function mapping Σ into the k-dimensional *force/torque space* as follows:

$$\mathbf{\Gamma}(\Sigma) = \{\mathbf{\Gamma}(\mathbf{r}) = [\mathbf{n}(\mathbf{r}), \mathbf{r} \times \mathbf{n}(\mathbf{r})] \mid \mathbf{r} \in \Sigma\} \subseteq \mathbb{R}^k, \tag{3}$$

where $\mathbf{r}$ denotes the position vector, and $\mathbf{n}(\mathbf{r})$, the unit normal to Σ at $\mathbf{r}$ pointing into K. $\mathbf{\Gamma}$ maps each point on Σ to a point $\mathbf{\Gamma}(\mathbf{r})$, corresponding to the force/torque resulting from a unit normal force exerted on K at $\mathbf{r}$.

If K is a two-dimensional object then $\mathbf{\Gamma}(\Sigma)$ is a one-dimensional curve in $\mathbb{R}^3$ and if K is three-dimensional object $\mathbf{\Gamma}(\Sigma)$ is a two-dimensional surface in $\mathbb{R}^6$.

We consider the surface Σ of an object K to be exceptional if positive grip on K is impossible. More precisely, we formulate the following

Definition 2.2 (Exceptional Surfaces) Let Σ be the boundary of an object K, where K is two-dimensional (respectively, three-dimensional). Σ is said to be an *exceptional surface* if for some force/torque vector $\mathbf{g} = [F, \tau]$ in $\mathbb{R}^3$ (respectively, $\mathbb{R}^6$) the following property holds:

> At each point $\mathbf{r}$ on Σ, the normal $\mathbf{n}(\mathbf{r})$ at $\mathbf{r}$ is perpendicular to $F + \tau \times \mathbf{r}$. $\square$

Lemma 2.1 *Let Σ be the boundary of an object K satisfying the assumptions 2.1. Then*

1. *The origin O lies in the convex-hull of $\mathbf{\Gamma}(\Sigma)$.*

2. *If, in addition, Σ is not an exceptional surface then the origin O lies in the interior of the convex-hull of $\mathbf{\Gamma}(\Sigma)$.*

PROOF.

(1) It suffices to show that

$$\left[\int_\Sigma \mathbf{n}(\mathbf{r})\, dS, \int_\Sigma (\mathbf{r} \times \mathbf{n}(\mathbf{r}))\, dS\right] = \mathbf{0}. \tag{4}$$

For each constant vector $\mathbf{c}$, we have

$$\mathbf{c} \cdot \int_\Sigma \mathbf{n}\, dS = \int_\Sigma \mathbf{c} \cdot \mathbf{n}\, dS = \int_K (\mathrm{div}\ \mathbf{c})\ dV = \mathbf{0} \tag{5}$$

by Gauss' divergence theorem, showing that $\int_\Sigma \mathbf{n}\, dS = \mathbf{0}$.

Similarly, for each constant vector $\mathbf{c}$, we have

$$\mathbf{c} \cdot \int_\Sigma \mathbf{r} \times \mathbf{n}\, dS = \int_\Sigma \mathbf{n} \cdot (\mathbf{c} \times \mathbf{r})\ dS = \int_K \text{div}\ (\mathbf{c} \times \mathbf{r})\ dV. \tag{6}$$

But div $(\mathbf{c} \times \mathbf{r}) = \mathbf{0}$. (For instance, if K is a three-dimensional object then $\mathbf{r} = (x, y, z)$, $\mathbf{c} = (c_1, c_2, c_3)$ and

$$\mathbf{c} \times \mathbf{r} = (c_2 z - c_3 y,\ c_3 x - c_1 z,\ c_1 y - c_2 x)\,.$$

Hence div $(\mathbf{c} \times \mathbf{r}) = \mathbf{0}$. A similar argument holds for the case when K is two-dimensional.) This shows that $\int_\Sigma \mathbf{r} \times \mathbf{n}\, dS = \mathbf{0}$, as claimed.

(2) Assume that Σ is not an exceptional surface. In consequence of the previous result, it is sufficient to show that the positive linear space generated by $\mathbf{\Gamma}(\Sigma)$ (denoted, pos $\mathbf{\Gamma}(\Sigma)$) spans the entire force/torque space.

Assume to the contrary. Then there is a vector $\mathbf{f} \notin$ pos $\mathbf{\Gamma}(\Sigma)$ in the force/torque space. We claim that $\mathbf{f}$ belongs to the linear subspace generated by $\mathbf{\Gamma}(\Sigma)$ (denoted, lin $\mathbf{\Gamma}(\Sigma)$). Otherwise, $\mathbf{f}$ will have a nonzero component $\mathbf{g} = [F, \tau]$ orthogonal to lin $\mathbf{\Gamma}(\Sigma)$. Then for each function $c(\mathbf{r})$ we have

$$\mathbf{g} \cdot \left[\int_\Sigma c(\mathbf{r})\mathbf{n}(\mathbf{r})\, dS, \int_\Sigma c(\mathbf{r})\ (\mathbf{r} \times \mathbf{n}(\mathbf{r}))\ dS\right] = 0,$$

or

$$\int_\Sigma c(\mathbf{r})\,[F \cdot \mathbf{n}(\mathbf{r}) + \tau \cdot (\mathbf{r} \times \mathbf{n}(\mathbf{r}))]\ dS = 0,$$

or

$$\int_\Sigma c(\mathbf{r})\ \ [(F + \tau \times \mathbf{r}) \cdot \mathbf{n}(\mathbf{r})]\ dS = 0. \tag{7}$$

In particular, substitute the function $c(\mathbf{r}) = (F + \tau \times \mathbf{r}) \cdot \mathbf{n}(\mathbf{r})$ into the last equation, to deduce that $(F + \tau \times \mathbf{r}) \cdot \mathbf{n}(\mathbf{r})$ must be identically 0 over Σ, contradicting the assumption that Σ is not exceptional.

Hence $\mathbf{f} \in$ lin $\mathbf{\Gamma}(\Sigma)$ and there exists a function (not necessarily nonnegative) $c'(\mathbf{r})$ such that

$$\left[\int_\Sigma c'(\mathbf{r})\mathbf{n}(\mathbf{r})\, dS, \int_\Sigma c'(\mathbf{r})(\mathbf{r} \times \mathbf{n}(\mathbf{r}))\, dS\right] = \mathbf{f}. \tag{8}$$

Now, suppose that $\gamma = \min_\Sigma c'(\mathbf{r}) < 0$. Then the function $c(\mathbf{r}) = c'(\mathbf{r}) - \gamma$ is everywhere nonnegative, and

$$\left[\int_\Sigma c(\mathbf{r})\mathbf{n}(\mathbf{r})\, dS,\ \int_\Sigma c(\mathbf{r})(\mathbf{r} \times \mathbf{n}(\mathbf{r}))\, dS\right] = \mathbf{f} \tag{9}$$

by the arguments used in the proof of (1). But this contradicts the assumption that $\mathbf{f} \notin \text{pos}\, \mathbf{\Gamma}(\Sigma)$. $\square$

Next we define the notion of a force/torque closure grasp; positive grips using such grasps are significant in various control algorithms for dexterous manipulation:

Definition 2.3 (Force/Torque Closure Grasp) A set of gripping points on an object K is said to constitute a *force/torque closure grasp* if and only if any arbitrary external force/torque acting on object K can be balanced by pressing the fingertips against this object at the selected grip points. $\square$

The following two classical theorems are stated without proof; the reader may consult the indicated references for their proofs.

Theorem 2.2 Carathéodory's Theorem. (See [5], [6]) *If* $X \subseteq \mathbb{R}^n$ *and* $p \in \text{ConvexHull}(X)$*, then* $p \in \text{ConvexHull}(Y)$ *for some* $Y \subseteq X$ *with* $|Y| \leq n+1$. $\square$

Theorem 2.3 Steinitz's Theorem. (See [6], [22], [23], [24]) *If* $X \subseteq \mathbb{R}^n$ *and* $p \in \text{Interior}(\text{ConvexHull}(X))$*, then* $p \in \text{Interior}(\text{ConvexHull}(Y))$ *for some* $Y \subseteq X$ *with* $|Y| \leq 2n$. $\square$

2.2 Existence Theorems

Theorem 2.4 *Let* K *be an object with boundary* Σ*, where* K *satisfies the assumptions 2.1. Then*

1. K *can be held at equilibrium by a positive grip of* $m \leq k+1$ *fingers; in particular*
$$m \leq \begin{cases} 4, & \textit{if } K \textit{ is two-dimensional;} \\ 7, & \textit{if } K \textit{ is three-dimensional.} \end{cases}$$

2. *If, in addition,* Σ *is not an exceptional surface then*

   (a) K *can be held with a force/torque closure grasp by a positive grip of* $m \leq 2k$ *fingers; in particular*
$$m \leq \begin{cases} 6, & \textit{if } K \textit{ is two-dimensional;} \\ 12, & \textit{if } K \textit{ is three-dimensional.} \end{cases}$$

(b) Any external force/torque $\mathbf{e} = [F, \tau]$ *acting on* K *can be balanced by a positive grip of* $m \leq k$ *fingers; in particular*

$$m \leq \begin{cases} 3, & \textit{if } K \textit{ is two-dimensional;} \\ 6, & \textit{if } K \textit{ is three-dimensional.} \end{cases}$$

PROOF.

(1) By the Lemma 2.1, the origin lies in the convex hull of $\mathbf{\Gamma}(\Sigma)$, *i.e.*,

$$\mathbf{0} \in \mathrm{ConvexHull}(\mathbf{\Gamma}(\Sigma)).$$

Carathéodory's Theorem implies that

$$\mathbf{0} \in \mathrm{ConvexHull}(Y),$$

for some $Y = \{\mathbf{\Gamma}(\mathbf{r}_1), \ldots, \mathbf{\Gamma}(\mathbf{r}_m)\} \subseteq \mathbf{\Gamma}(\Sigma)$, and $m \leq k+1$. If K is two-dimensional, then $\mathbf{\Gamma}(\Sigma) \subseteq \mathbb{R}^3$ and $m \leq 4$; and if K is three-dimensional, then $\mathbf{\Gamma}(\Sigma) \subseteq \mathbb{R}^6$ and $m \leq 7$.

Hence there exist positive real quantities $\alpha_1, \ldots, \alpha_m$ with $\alpha_1 + \cdots + \alpha_m = 1$ such that

$$\alpha_1 \, \mathbf{\Gamma}(\mathbf{r}_1) + \cdots + \alpha_m \, \mathbf{\Gamma}(\mathbf{r}_m) = \mathbf{0}. \tag{10}$$

The positive grip consists of the grip points $\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ and the force targets $\{\alpha_1, \ldots, \alpha_m\}$.

($2a$) By Lemma 2.1, the origin lies in the interior of the convex-hull of $\mathbf{\Gamma}(\Sigma)$, i.e.

$$\mathbf{0} \in \mathrm{Interior}(\mathrm{ConvexHull}(\mathbf{\Gamma}(\Sigma))).$$

Steinitz's Theorem implies that

$$\mathbf{0} \in \mathrm{Interior}(\mathrm{ConvexHull}(Y)),$$

for some $Y = \{\mathbf{\Gamma}(\mathbf{r}_1), \ldots, \mathbf{\Gamma}(\mathbf{r}_m)\} \subseteq \mathbf{\Gamma}(\Sigma)$ and $m \leq 2k$. If K is two-dimensional, then $\mathbf{\Gamma}(\Sigma) \subseteq \mathbb{R}^3$ and $m \leq 6$; and if K is three-dimensional, then $\mathbf{\Gamma}(\Sigma) \subseteq \mathbb{R}^6$ and $m \leq 12$.

It is easy to see that pos Y spans the entire force/torque space; hence, for every external force/torque $\mathbf{e} = [F, \tau]$, it is possible to express $-\mathbf{e}$ as a positive linear combination of

$$\mathbf{\Gamma}(\mathbf{r}_1), \ldots, \mathbf{\Gamma}(\mathbf{r}_m).$$

As a result, every external force/torque can be balanced by a set of normal forces of appropriate magnitudes, applied at the grip points $\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$.

(2*b*) By Lemma 2.1, the origin lies in the interior of the convex-hull of $\mathbf{\Gamma}(\Sigma)$, i.e.

$$\mathbf{0} \in \text{Interior}(\text{ConvexHull}(\mathbf{\Gamma}(\Sigma))).$$

Hence a k-dimensional ball of radius ε centered around the origin is contained in the convex hull of $\mathbf{\Gamma}(\Sigma)$.

Let $L^+ = \{\lambda(-\mathbf{e}) \mid \lambda > 0\}$ be the positive ray in the direction opposite of $\mathbf{e}$, let $\mathbf{p} = \lambda_0(-\mathbf{e})$ $(\lambda_0 > 0)$ be the position vector corresponding to the boundary point p of ConvexHull($\mathbf{\Gamma}(\Sigma)$) on L^+, and let H be a $(k-1)$-dimensional hyperplane which supports the ConvexHull($\mathbf{\Gamma}(\Sigma)$) at p. Clearly $\mathbf{p} \in \text{ConvexHull}(\mathbf{\Gamma}(\Sigma) \cap H)$. By Carathéodory's Theorem,

$$\mathbf{p} \in \text{ConvexHull}(Y),$$

for some $Y = \{\mathbf{\Gamma}(\mathbf{r}_1), \ldots, \mathbf{\Gamma}(\mathbf{r}_m)\} \subseteq \mathbf{\Gamma}(\Sigma) \cap H$, and $m \leq k$. If K is two-dimensional, then $\mathbf{\Gamma}(\Sigma) \subseteq \mathbb{R}^3$ and $m \leq 3$; and if K is three-dimensional, then $\mathbf{\Gamma}(\Sigma) \subseteq \mathbb{R}^6$ and $m \leq 6$.

As before, there exist positive real quantities $\alpha_1, \ldots, \alpha_m$ with $\alpha_1 + \cdots + \alpha_m = 1$ such that

$$\alpha_1 \mathbf{\Gamma}(\mathbf{r}_1) + \cdots + \alpha_m \mathbf{\Gamma}(\mathbf{r}_m) = \mathbf{p} = \lambda_0(-\mathbf{e}),$$

or

$$\mathbf{e} + \alpha'_1 f\mathbf{\Gamma}(\mathbf{r}_1) + \cdots + \alpha'_m \mathbf{\Gamma}(\mathbf{r}_m) = \mathbf{0}, \tag{11}$$

where

$$\alpha'_i = \frac{\alpha_i}{\lambda_0}, \quad 1 \leq i \leq m.$$

The positive grip consists of the grip points $\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ and the force targets $\{\alpha'_1, \ldots, \alpha'_m\}$. $\square$

Remark 2.4 This result can be strengthened to imply the existence of a positive grip with three fingers to hold any arbitrary two-dimensional object K at equilibrium. Let the object K be held at equilibrium with four fingers. Let $\mathbf{r}_1, \ldots, \mathbf{r}_4$ be four grip points on Σ which yield a positive grip. Vary $\mathbf{r}_4$ continuously along Σ in some direction until it coincides with one of the other three $\mathbf{r}_i$'s. Suppose without loss of generality that $\mathbf{r}_4$ coincides with $\mathbf{r}_1$. If O lies in the convex hull of the three points $\mathbf{\Gamma}(\mathbf{r}_1)$, $\mathbf{\Gamma}(\mathbf{r}_2)$, $\mathbf{\Gamma}(\mathbf{r}_3) \in \mathbf{\Gamma}(\Sigma)$, then the claim is obviously true. Otherwise, there will be a critical value of $\mathbf{r}_4$ at which the origin reaches the boundary of the convex hull of the points $\mathbf{\Gamma}(\mathbf{r}_i)$, $i = 1, \ldots, 4$. It is plain that at this value of $\mathbf{r}_4$ the origin is a convex combination of $\mathbf{\Gamma}(\mathbf{r}_4)$ and of two of the remaining $\mathbf{\Gamma}(\mathbf{r}_i)$'s, which again establishes the claim.

In the case K is three-dimensional, we believe that a similar argument will show that one can always obtain a positive grip on K using only five fingers; here, we expect to reduce the number of grip points by two because $\mathbf{\Gamma}(\Sigma)$ is a two-dimensional manifold, allowing us to reach a face of codimension two (rather than one) on the convex hull of the seven points $\mathbf{\Gamma}(\mathbf{r}_i)$.

However these degenerate grips are not secure, in the sense that a small perturbation of the directions of the forces being applied will cause the fingers to slip, if there is no static friction at the contact points. This makes these grips undesirable in practice. $\square$

2.3 Exceptional Surfaces

In this subsection, we analyze the shape of the exceptional surfaces (Cf. Definition 2.2).

Suppose that the surface Σ is exceptional, let F and τ be as in Definition 2.2 and consider the vector field defined by $\dot{\mathbf{r}} = F + \tau \times \mathbf{r}$. For each point $\mathbf{r}$ let $\gamma_{\mathbf{r}}$ be the curve satisfying this equation and passing through $\mathbf{r}$. Clearly if $\mathbf{r}$ lies on the boundary Σ of K then Σ must contain the entire curve $\gamma_{\mathbf{r}}$. Thus Σ must be a surface generated by a family of such curves.

To obtain an explicit form for these curves, assume without loss of generality that $\tau = \mathbf{k}$, and write $F = (F_1, F_2, F_3)$. The differential equation for the curves γ then becomes

$$\begin{aligned} \dot{x} &= F_1 - y \\ \dot{y} &= F_2 + x \\ \dot{z} &= F_3 \end{aligned} \tag{12}$$

whose solution is easily seen to be

$$\mathbf{r} = ((-F_2 + c_1 \cos t), (F_1 + c_1 \sin t), (F_3 t + c_2)) \tag{13}$$

That is, $\gamma_{\mathbf{r}}$ is either a straight line, a circle, or a helix. Since Σ is assumed to be bounded, only circles are possible. It follows from the above equation that all the circles γ must be parallel with their centers lying on a line perpendicular to their planes, and thus Σ must be a (bounded) surface of revolution. Conversely, it is easily seen that if Σ is a surface of revolution it is indeed impossible to balance every possible force and torque by gripping the body. For example, if Σ is a sphere, then any normal force applied to it will have zero torque relative to its center, and thus no non-zero torque can be resisted, no matter how many (perfectly frictionless) fingers grip the sphere. More generally, given any surface of revolution Σ, no torque in the direction of its axis can be resisted by frictionless gripping.

It is also worth noting that if unbounded objects were also allowed, the class of exceptional surfaces would also admit (infinite) cylinders (of arbitrary cross-section) and helical surfaces as exceptional surfaces. It is clear that each such surface admits some specific external force/torque pattern which no set of frictionless gripping fingers can resist.

Of course, in all of our exceptional cases the force/torque patterns which cannot be resisted only move Σ between indistinguishable positions, at least if Σ is not assumed to possess any additional significant feature. That is, any 2-dimensional (respectively, 3-dimensional) object K whatsoever can always be held by four (resp. seven) fingers in a frictionless grip which resists any external force and torque on K which tends to move it to a significantly different spatial position.

3 Synthesis of Positive Grips

Let K be a given smooth object satisfying the assumptions 2.1 of the previous section. It has been shown that K can always be gripped using only a certain specific number of fingers, this number depending solely on what type of grasp is desired and whether K is two- or three-dimensional.

In this section, we present an efficient algorithm to synthesize such positive grips in the special case when K is a bounded polyhedron (resp. polygon) with n faces (resp. sides) and the shape of the object is known precisely. If K is a polyhedron, we also assume that a triangulation of each of the faces is also provided. The Grip-Synthesis algorithm presented has a time complexity of $O(n)$, when the dimension of K is fixed, and is therefore quite efficient enough for practical applications. This algorithm may also be used for more general objects, for which a polyhedral approximation is available.

We present our grip synthesis procedure in two parts:

- First, we show how to synthesize a positive grip for K using $O(n)$ fingers in $O(n)$ time;

- Then we describe how to reduce this grip to another with the bounded number of fingers found in the previous section; in each case, the algorithm for the reduction step takes $O(n)$ time.

3.1 Preliminary Results: Synthesis of Positive Grips using $O(n)$ fingers

Theorem 3.1 *Let K be a bounded polyhedral object with n faces. Then*

1. *K can be held at equilibrium by a positive grip using $O(n)$ fingers.*
2. *K can be held with a force/torque closure grasp by a positive grip using $O(n)$ fingers.*
3. *Any external force/torque $\mathbf{e} = [F, \tau]$ acting on K can be balanced by a positive grip using $O(n)$ fingers.*

All these positive grips can be synthesized in $O(n)$ time.

PROOF.
We describe the proof in detail for the three-dimensional case, and comment briefly on the similar and simpler arguments needed for the two-dimensional case.

Let t_1, t_2, ..., t_N be the set of triangular regions on the surface of K, given by the triangulation of its faces. Since at each vertex of K three or more edges meet, by Euler's Theorem, the number of edges of K is bounded from above by $3n$; and since the number of triangles in a triangulation of a face is exactly one less than the number of edges of that face, it follows that $N \leq 6n = O(n)$.

Since the polyhedral object K is bounded, all its faces are bounded and K has a vertex v where three or more faces meet. Let t_1, t_2 and t_3 be some three triangular regions meeting at v, and lying on three distinct faces (that is, t_1, t_2 and t_3 are pair-wise noncoplanar). Divide each of the triangular regions t_i ($i = 1, 2, 3$) into three triangular subregions t_i', t_i'' and t_i''' whose centroids are not collinear [1].

[1] Consider a triangle ABC with centroid G. Divide ABC into three triangular regions AGB, BGC and CGA, whose centroids are respectively X, Y and Z. It can be shown that $XY \| CA$, $YZ \| AB$ and $ZX \| BC$, and the triangle XYZ is similar to ABC. Hence X, Y and Z are not collinear.

Consider the following set of $N+6$ grip points:

$$\bigcup_{i=1,2,3} \{\mathrm{Centroid}(t'_i), \mathrm{Centroid}(t''_i), \mathrm{Centroid}(t'''_i)\} \cup \bigcup_{4\le i\le N} \{\mathrm{Centroid}(t_i)\}\,. \tag{14}$$

Enumerate these points as $\mathbf{r}_i$ and let $\mathbf{\Gamma}(\mathbf{r}_i) = [\mathbf{n}(\mathbf{r}_i), \mathbf{r}_i \times \mathbf{n}(\mathbf{r}_i)]$ be the force/torque vector resulting from a unit normal force pointing into K at $\mathbf{r}_i$.

(1) Define a corresponding set of area values α_i by

$$\bigcup_{i=1,2,3} \{\mathrm{Area}(t'_i), \mathrm{Area}(t''_i), \mathrm{Area}(t'''_i)\} \cup \bigcup_{4\le i\le N} \{\mathrm{Area}(t_i)\}\,, \tag{15}$$

Then it follows as in the proof of the first part of the Lemma 2.1 that

$$\sum_{i=1}^{N+6} \alpha_i\, \mathbf{\Gamma}(\mathbf{r}_i) = \mathbf{0} \tag{16}$$

Hence K can be held at equilibrium by the positive grip consisting of the grip points $\{\mathbf{r}_1, \ldots, \mathbf{r}_{N+6}\}$ and the force targets $\{\alpha_1, \ldots, \alpha_{N+6}\}$. It is easy to see that this positive grip can be synthesized in $O(n)$ time.

(2) Next we show that the positive space spanned by the $\mathbf{\Gamma}(\mathbf{r}_i)$'s, *i.e.*,,

$$\mathrm{pos}\ (\mathbf{\Gamma}(\mathbf{r}_1), \ldots, \mathbf{\Gamma}(\mathbf{r}_{N+6}))$$

spans the entire force/torque space. This will imply that the same positive grip achieves a force/torque closure grasp.

Assume to the contrary; then there is a vector $\mathbf{f} \notin \mathrm{pos}\ (\mathbf{\Gamma}(\mathbf{r}_1), \ldots, \mathbf{\Gamma}(\mathbf{r}_{N+6}))$ in the force/torque space. We claim that $\mathbf{f}$ belongs to the linear space spanned by $\mathbf{\Gamma}(\mathbf{r}_1), \ldots, \mathbf{\Gamma}(\mathbf{r}_9)$. Indeed, in the contrary case, $\mathbf{f}$ will have a nonzero component $\mathbf{g} = [F, \tau]$ orthogonal to

$$\mathrm{lin}\ (\mathbf{\Gamma}(\mathbf{r}_1), \ldots, \mathbf{\Gamma}(\mathbf{r}_9))\,.$$

so that for every choice of $\beta_1, \ldots, \beta_9$,

$$\mathbf{g} \cdot \left[\sum_{i=1}^{9} \beta_i\, \mathbf{n}(\mathbf{r}_i),\, \sum_{i=1}^{9} \beta_i\, (\mathbf{r}_i \times \mathbf{n}(\mathbf{r}_i)) \right] = 0,$$

i.e.

$$\sum_{i=1}^{9} \beta_i\, [F \cdot \mathbf{n}(\mathbf{r}_i) + \tau \cdot (\mathbf{r}_i \times \mathbf{n}(\mathbf{r}_i))] = 0,$$

so that

$$\sum_{i=1}^{9} \beta_i \left[(F + \tau \times \mathbf{r}_i) \cdot \mathbf{n}(\mathbf{r}_i) \right] = 0. \tag{17}$$

But then we can substitute the values $\beta_i = (F + \tau \times \mathbf{r}_i) \cdot \mathbf{n}(\mathbf{r}_i)$ into the equation (17), and deduce that

$$(F + \tau \times \mathbf{r}_i) \cdot \mathbf{n}(\mathbf{r}_i) = 0, \quad \text{for } i = 1, \ldots, 9. \tag{18}$$

Let $i \in \{1, 2, 3\}$. Since $\mathbf{n}(\mathbf{r}_{3i-2}) = \mathbf{n}(\mathbf{r}_{3i-1}) = \mathbf{n}(\mathbf{r}_{3i})$, equation (18) implies that

$$\begin{aligned} -F \cdot \mathbf{n}(\mathbf{r}_{3i-2}) &= (\tau \times \mathbf{r}_{3i-2}) \cdot \mathbf{n}(\mathbf{r}_{3i-2}) \\ &= (\tau \times \mathbf{r}_{3i-1}) \cdot \mathbf{n}(\mathbf{r}_{3i-2}) \\ &= (\tau \times \mathbf{r}_{3i}) \cdot \mathbf{n}(\mathbf{r}_{3i-2}) \end{aligned}$$

Hence

$$\tau \cdot \left[(\mathbf{r}_{3i-2} - \mathbf{r}_{3i}) \times \mathbf{n}(\mathbf{r}_{3i-2}) \right] = 0 = \tau \cdot \left[(\mathbf{r}_{3i-1} - \mathbf{r}_{3i}) \times \mathbf{n}(\mathbf{r}_{3i-2}) \right]. \tag{19}$$

and since the vectors $\mathbf{r}_{3i-2} - \mathbf{r}_{3i}$ and $\mathbf{r}_{3i-1} - \mathbf{r}_{3i}$ are linearly independent it follows that τ is orthogonal to two nonzero vectors which lie in the plane of t_i. Hence τ is orthogonal to each of the planes of t_1, t_2 and t_3, so that τ must be zero. Moreover, equation (18) tells us that

$$F \cdot \mathbf{n}(\mathbf{r}_{3i-2}) = 0, \quad i \in \{1, 2, 3\}.$$

Hence F is parallel to each of the planes of t_1, t_2 and t_3, so that F must be zero also. Hence $\mathbf{g} = \mathbf{0}$, a contradiction proving that the linear space spanned by $\mathbf{\Gamma}(\mathbf{r}_i)$'s is the entire force/torque space as asserted.

It follows that $\mathbf{f} \in \text{lin } (\mathbf{\Gamma}(\mathbf{r}_1), \ldots, \mathbf{\Gamma}(\mathbf{r}_9))$, so there exist scalar quantities $\beta_1, \ldots, \beta_9$ (not necessarily all nonnegative) such that

$$\sum_{i=1}^{9} \beta_i \, \mathbf{\Gamma}(\mathbf{r}_i) = \mathbf{f}. \tag{20}$$

Put

$$\gamma = \min \left[0, \min_{1 \le i \le 9} \left(\frac{\beta_i}{\alpha_i} \right) \right] \le 0, \tag{21}$$

and let

$$\begin{aligned} \alpha_i' &= -\gamma \cdot \alpha_i + \beta_i, \qquad && i = 1, \ldots, 9, \\ \alpha_i' &= -\gamma \cdot \alpha_i, && i = 10, \ldots, N + 6. \end{aligned} \tag{22}$$

Then all the α_i''s are nonnegative and

$$\sum_{i=1}^{N+6} \alpha_i' \, \mathbf{\Gamma}(\mathbf{r}_i) = (-\gamma) \sum_{i=1}^{N+6} \alpha_i \, \mathbf{\Gamma}(\mathbf{r}_i) + \sum_{i=1}^{9} \beta_i \, \mathbf{\Gamma}(\mathbf{r}_i) = \mathbf{f}. \tag{23}$$

contradicting the assumption that $\mathbf{f} \notin \text{pos } (\mathbf{\Gamma}(\mathbf{r}_1), \ldots, \mathbf{\Gamma}(\mathbf{r}_{N+6}))$.

(3) Let $\mathbf{e} = [F, \tau]$ be a given external force/torque and $\mathbf{f} = -\mathbf{e}$. From the above arguments, we know that

$$\mathbf{f} \in \text{lin } (\mathbf{\Gamma}(\mathbf{r}_1), \ldots, \mathbf{\Gamma}(\mathbf{r}_9)),$$

so that there exist β_i's (not necessarily all nonnegative) such that

$$\sum_{i=1}^{9} \beta_i \, \mathbf{\Gamma}(\mathbf{r}_i) = \mathbf{f}. \tag{24}$$

If we choose α_i''s as in equation (22) then

$$\sum_{i=1}^{N+6} \alpha_i' \, \mathbf{\Gamma}(\mathbf{r}_i) = \sum_{i=1}^{N+6} -\gamma \alpha_i \, \mathbf{\Gamma}(\mathbf{r}_i) + \sum_{i=1}^{9} \beta_i \, \mathbf{\Gamma}(\mathbf{r}_i) = \mathbf{f} = -\mathbf{e}, \tag{25}$$

where γ is given by the equation (21). Hence, the positive grip consisting of the grip points $\{\mathbf{r}_1, \ldots, \mathbf{r}_{N+6}\}$ and the force targets $\{\alpha_1', \ldots, \alpha_{N+6}'\}$ will resist $\mathbf{e}$. Since the β_i's can be found in $O(1)$ time, it follows that the force targets α_i' can also be computed in $O(n)$ time.

If K is a two-dimensional object then the above arguments apply with the following easy modifications. Since the polygonal object K is bounded, all its sides are bounded and K has a vertex v at which two sides, t_1 and t_2 meet. Let $t_3, \ldots, t_n$ be the remaining sides. Divide the side t_1 into two segments t_1' and t_1''.

Consider the following set of $(n+1)$ grip points:

$$\{\text{Center}(t_1'), \text{Center}(t_1'')\} \cup \bigcup_{2 \le i \le n} \{\text{Center}(t_i)\}. \tag{26}$$

Enumerate these points as $\mathbf{r}_i$ and let $\mathbf{\Gamma}(\mathbf{r}_i)$ be the associate force/torque vectors. Define a corresponding set of length values α_i's by

$$\{\text{Length}(t_i'), \text{Length}(t_i'')\} \cup \bigcup_{2 \le i \le n} \{\text{Length}(t_i)\}. \tag{27}$$

Much as in the preceding proof, we can show that

- $$\sum_{i=1}^{n+1} \alpha_i \, \mathbf{\Gamma}(\mathbf{r}_i) = \mathbf{0}.$$

- lin $(\mathbf{\Gamma}(\mathbf{r}_1), \mathbf{\Gamma}(\mathbf{r}_2), \mathbf{\Gamma}(\mathbf{r}_3))$ and thus also pos $(\mathbf{\Gamma}(\mathbf{r}_1), \ldots, \mathbf{\Gamma}(\mathbf{r}_{n+1}))$ is the entire force/torque space.

- Positive grips using an appropriate subset of these grip points can be synthesized in $O(n)$ time. $\square$

3.2 Reduction Algorithms

In this subsection, we show how to reduce a positive grip using $l = O(n)$ fingers to another involving only m fingers, where m matches the bounds found in section 2. The algorithm follows Gale's proof of Steinitz's Theorem (see [8]) closely and is based on the following main idea:

Assume that

$$\alpha_1 \mathbf{v}_1 + \cdots \alpha_l \mathbf{v}_l = \alpha \, \mathbf{v}, \tag{28}$$

where the α_i's and α are positive scalar quantities and the $\mathbf{v}_i$'s and $\mathbf{v}$ are k-dimensional non-null vectors. Our aim is to find $m \le k$ vectors

$$\{\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_m}\} \subseteq \{\mathbf{v}_1, \ldots, \mathbf{v}_l\}$$

and positive scalar quantities α'_1, …, α'_m and α' such that

$$\alpha'_1 \mathbf{v}_{i_1} + \cdots \alpha'_m \mathbf{v}_{i_m} = \alpha' \, \mathbf{v}. \tag{29}$$

If $l \le k$ there is nothing to do. Otherwise, choose k vectors among the $\mathbf{v}_i$'s, say the first k. There are two cases to consider:

- Case 1 $\mathbf{v}_1$, …, $\mathbf{v}_k$ *are linearly dependent.*
  In this case there exist β_1, …, β_k not all 0, such that

  $$\beta_1 \mathbf{v}_1 + \cdots + \beta_k \mathbf{v}_k = \mathbf{0}. \tag{30}$$

  Reversing the signs of all the β_i, if necessary, we can assume that at least one β_i is negative and non zero.

  Now put

  $$\gamma = \max_{\beta_i < 0} \left(\frac{\alpha_i}{\beta_i} \right) < 0;$$

for specificity, suppose that $\gamma = \dfrac{\alpha_1}{\beta_1}$. Put $\alpha_i' = \alpha_i - \gamma\beta_i$, for $1 \le i \le k$. Then

$$\alpha_1'\mathbf{v}_1 + \cdots + \alpha_k'\mathbf{v}_k + \alpha_{k+1}\mathbf{v}_{k+1} + \cdots \alpha_l\mathbf{v}_l = \alpha\,\mathbf{v}. \tag{31}$$

Since $\alpha_1' = 0$ and α_2', ..., $\alpha_k' \ge 0$, we have reduced the number of vectors by at least 1 in this case.

- CASE 2 $\mathbf{v}_1$, ..., $\mathbf{v}_k$ *are linearly independent.*
  In this case, there exist β_1, ..., β_k such that

$$\beta_1\mathbf{v}_1 + \cdots + \beta_k\mathbf{v}_k = \mathbf{v}. \tag{32}$$

Assume that at least one $\beta_i < 0$, since otherwise we have found the desired $\alpha_i' = \beta_i$. Put

$$\gamma = \max_{\beta_i < 0}\left(\frac{\alpha_i}{\beta_i}\right) < 0$$

and for specificity, suppose that $\gamma = \dfrac{\alpha_1}{\beta_1}$. Put $\alpha_i' = \alpha_i - \gamma\beta_i$, for $1 \le i \le k$. Then

$$\alpha_1'\mathbf{v}_1 + \cdots + \alpha_k'\mathbf{v}_k + \alpha_{k+1}\mathbf{v}_{k+1} + \cdots \alpha_l\mathbf{v}_l = (\alpha - \gamma)\,\mathbf{v}. \tag{33}$$

Since $\alpha_1' = 0$, α_2', ..., $\alpha_k' \ge 0$, and $(\alpha - \gamma) > 0$, we have reduced the number of vectors by at least 1 in this case also.

Hence, repeating the above steps at most $(l-k)$-times, we find α_i', $i = 1$, ..., $m < k$, with the desired properties. Since each basic step only involves solution of a linear system involving k k-dimensional vectors and modification of $O(k)$ coefficients, each such step takes $O(1)$ time (i.e., independent of l). Hence the entire algorithm takes $O(l)$ time.

As a corollary of the above arguments we see that

Lemma 3.2 *Let* $\mathbf{v}$ *be a positive linear combination of* $l = O(n)$ *vectors* $\mathbf{v}_1$, ..., $\mathbf{v}_l$, *i.e.,*

$$\alpha_1\,\mathbf{v}_1 + \cdots + \alpha_l\,\mathbf{v}_l = \mathbf{v},$$

where $\mathbf{v}$*'s are* k*-dimensional non-null vectors. Then* $\mathbf{v}$ *can be expressed as a positive linear combination of at most* k *of the vectors* $\mathbf{v}_i$*; and the coefficients which achieve this can be computed in* $O(n)$ *time.* □

Theorem 3.3 *Let* K *be a bounded polyhedral/polygonal object with* n *faces/sides. Then*

1. *K can be held at equilibrium by a positive grip using $m \le k+1$ fingers, i.e.*
$$m \le \begin{cases} 4, & \textit{if } K \textit{ is two-dimensional;} \\ 7, & \textit{if } K \textit{ is three-dimensional.} \end{cases}$$

2. *K can be held with a force/torque closure grasp by a positive grip using $m \le 2k$ fingers, i.e.*
$$m \le \begin{cases} 6, & \textit{if } K \textit{ is two-dimensional;} \\ 12, & \textit{if } K \textit{ is three-dimensional.} \end{cases}$$

3. *Any external force/torque $\mathbf{e} = [F, \tau]$ acting on K can be balanced by a positive grip using $m \le k$ fingers, i.e.*
$$m \le \begin{cases} 3, & \textit{if } K \textit{ is two-dimensional;} \\ 6, & \textit{if } K \textit{ is three-dimensional.} \end{cases}$$

Each of these positive grips can be synthesized in $O(n)$ time. PROOF.

(1) We know from Theorem 3.1 that K can be held at equilibrium by a positive grip using $O(n)$ fingers with grip points $\{\mathbf{r}_1, \ldots, \mathbf{r}_l\}$ and force targets $\{\alpha_1, \ldots, \alpha_l\}$ which can be found in $O(n)$ time and satisfy

$$\alpha_1 \boldsymbol{\Gamma}(\mathbf{r}_1) + \cdots + \alpha_l \boldsymbol{\Gamma}(\mathbf{r}_l) = \mathbf{0}. \tag{34}$$

Let $\mathbf{v} = -\boldsymbol{\Gamma}(\mathbf{r}_l)$. Then $\mathbf{v}$ can be expressed as a positive linear combination of $\boldsymbol{\Gamma}(\mathbf{r}_1), \ldots, \boldsymbol{\Gamma}(\mathbf{r}_{l-1})$, i.e.,

$$\mathbf{v} = \left(\frac{\alpha_1}{\alpha_l}\right) \boldsymbol{\Gamma}(\mathbf{r}_1) + \cdots + \left(\frac{\alpha_{l-1}}{\alpha_l}\right) \boldsymbol{\Gamma}(\mathbf{r}_{l-1}). \tag{35}$$

By the Lemma 3.2, the above equation can be transformed to an equation of the form
$$\mathbf{v} = \alpha_1' \boldsymbol{\Gamma}(\mathbf{r}_{i_1}) + \cdots + \alpha_m' \boldsymbol{\Gamma}(\mathbf{r}_{i_m}), \quad \text{where } m \le k,$$
in $O(n)$ time. Hence

$$\alpha_1' \boldsymbol{\Gamma}(\mathbf{r}_{i_1}) + \cdots + \alpha_m' \boldsymbol{\Gamma}(\mathbf{r}_{i_m}) + \boldsymbol{\Gamma}(\mathbf{r}_l) = \mathbf{0}. \tag{36}$$

Thus we can take the required positive grip to consist of grip points $\{\mathbf{r}_{i_1}, \ldots, \mathbf{r}_{i_m}, \mathbf{r}_l\}$, with the force targets $\{\alpha_1', \ldots, \alpha_m', 1\}$.

(2) We know from Theorem 3.1 that K can be held with a force/torque closure grasp by a positive grip involving $O(n)$ fingers, consisting of grip points $\{\mathbf{r}_1, \ldots, \mathbf{r}_l\}$, i.e.

$$\text{pos } (\mathbf{\Gamma}(\mathbf{r}_1), \ldots, \mathbf{\Gamma}(\mathbf{r}_l)) = \mathbb{R}^k,$$

and that this grip can be found in $O(n)$ time. Choose the first k grip points so that

$$\text{lin } (\mathbf{\Gamma}(\mathbf{r}_1), \ldots, \mathbf{\Gamma}(\mathbf{r}_k)) = \mathbb{R}^k,$$

and let

$$\mathbf{v} = -\left(\mathbf{\Gamma}(\mathbf{r}_1) + \cdots + \mathbf{\Gamma}(\mathbf{r}_k)\right). \tag{37}$$

Then $\mathbf{v}$ can be expressed as a positive linear combination of $\mathbf{\Gamma}(\mathbf{r}_1), \ldots, \mathbf{\Gamma}(\mathbf{r}_l)$, i.e.

$$\mathbf{v} = \alpha_1 \mathbf{\Gamma}(\mathbf{r}_1) + \cdots + \alpha_l \mathbf{\Gamma}(\mathbf{r}_l), \tag{38}$$

where the coefficients α_i can be calculated in $O(n)$ time.

By Lemma 3.2, the above equation can be transformed to an equation of the form

$$\mathbf{v} = \alpha_1' \mathbf{\Gamma}(\mathbf{r}_{i_1}) + \cdots + \alpha_m' \mathbf{\Gamma}(\mathbf{r}_{i_m}), \quad \text{where } m \leq k, \tag{39}$$

in $O(n)$ time.

We claim that the required positive grip can be taken to involve no more than $m + k$ $(\leq 2k)$ grip points

$$\{\mathbf{r}_1, \ldots, \mathbf{r}_k\} \cup \{\mathbf{r}_{i_1}, \ldots, \mathbf{r}_{i_m}\}$$

of the total set with which we began. This is a consequence of the fact that

$$\text{pos } (\{\mathbf{\Gamma}(\mathbf{r}_1), \ldots, \mathbf{\Gamma}(\mathbf{r}_k)\} \cup \{\mathbf{\Gamma}(\mathbf{r}_{i_1}), \ldots, \mathbf{\Gamma}(\mathbf{r}_{i_m})\}) = \mathbb{R}^k. \tag{40}$$

Indeed let $\mathbf{f}$ be a vector in the force/torque space. Since the linear subspace generated by $\mathbf{\Gamma}(\mathbf{r}_1), \ldots, \mathbf{\Gamma}(\mathbf{r}_k)$ is the entire force/torque space,

$$\beta_1 \mathbf{\Gamma}(\mathbf{r}_1) + \cdots + \beta_k \mathbf{\Gamma}(\mathbf{r}_k) = \mathbf{f}. \tag{41}$$

If β_i's are all nonnegative then there is nothing to prove. If not, put

$$\gamma = \min_{1 \leq i \leq k} \beta_i < 0.$$

Then

$$\begin{aligned}
\sum_{i=1}^{k}(\beta_i-\gamma)\mathbf{\Gamma}(\mathbf{r}_i)+\sum_{j=1}^{m}(-\gamma\alpha'_j)\mathbf{\Gamma}(\mathbf{r}_{i_j}) &= \sum_{i=1}^{k}(\beta_i-\gamma)\mathbf{\Gamma}(\mathbf{r}_i)+(-\gamma)\mathbf{v} \\
&= \sum_{i=1}^{k}(\beta_i-\gamma)\mathbf{\Gamma}(\mathbf{r}_i)+(-\gamma)\sum_{i=1}^{k}-\mathbf{\Gamma}(\mathbf{r}_i) \\
&= \sum_{i=1}^{k}\beta_i\mathbf{\Gamma}(\mathbf{r}_i)=\mathbf{f}
\end{aligned}$$

Since $(\beta_i-\gamma)\geq 0$ and $(-\gamma\alpha'_j)>0$, (40) follows at once.

(3) We know from Theorem 3.1 that an external force/torque $\mathbf{e}=[F,\tau]$ acting on K can be balanced by a positive grip using $O(n)$ fingers with grip points $\{\mathbf{r}_1, \ldots, \mathbf{r}_l\}$ and force targets $\{\alpha_1, \ldots, \alpha_l\}$, i.e.

$$\alpha_1\,\mathbf{\Gamma}(\mathbf{r}_1)+\cdots+\alpha_l\,\mathbf{\Gamma}(\mathbf{r}_l)=-\mathbf{e}; \tag{42}$$

moreover, this grip can be found in $O(n)$ time.

Using Lemma 3.2, this equation can be transformed to an equation of the form

$$\alpha'_1\mathbf{\Gamma}(\mathbf{r}_{i_1})+\cdots+\alpha'_m\mathbf{\Gamma}(\mathbf{r}_{i_m})=-\mathbf{e},\quad \text{where } m\leq k,$$

in $O(n)$ time. Hence

$$\mathbf{e}+\alpha'_1\mathbf{\Gamma}(\mathbf{r}_{i_1})+\cdots+\alpha'_m\mathbf{\Gamma}(\mathbf{r}_{i_m})=\mathbf{0}, \tag{43}$$

and the positive grip which we require involves the grip points $\{\mathbf{r}_{i_1}, \ldots, \mathbf{r}_{i_m}\}$, and the force targets $\{\alpha'_1, \ldots, \alpha'_m\}$. $\square$

4 Concluding Remarks

Several problems related to those we have discussed remain to be explored. Certain applications may require that the fingers be placed only in certain permissible regions on the surface of the body being grasped, or that the force targets may remain within some prescribed intervals. Moreover our assumption that the body to be gripped is rigid is rather restrictive, since in practice robot hands will often be required to grip soft or pliable objects, or tools with hinges and movable parts. The algorithms presented here are not directly applicable in such situations, and generalized algorithms need to be sought.

The presence of static friction between a gripped object and a set of gripping fingers will allow grips with smaller number of fingers; algorithms to synthesize such grips may differ significantly from those we have presented and need to be investigated (see [21] for some initial results on the synthesis of frictional grips).

We have begun to study this entire complex of problems and plan to use the resulting set of algorithms in work with an experimental manipulator. We hope that this experimental work will reveal other questions worth studying.

5 Acknowledgement

We wish to acknowledge with gratitude the rewarding conversations with Prof. Ravi Kannan of C-M.U. and Prof. Alan Siegel and Dr. Gerardo Lafferriere of N.Y.U.

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