On the Other Hands:
Geometric Ideas in Robotics

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1 Truth About Polydactility

I would like to start this essay with the following story based on an essay entitled "Eight Little Piggies" by the popular Harvard Biologist Stephen Jay Gould [Gou93].

The fossils of the oldest tetrapods were discovered in eastern Greenland by a Danish expedition in 1929. They date back to the very last phase of the Devonian period (the so-called "age-of-fish") and the two genera from that period that have been studied extensively are *Ichthyostega* and *Acanthostega*. The Swedish paleontologist Gunnar Säve-Söderbergh collected most of the material in 1931 and directed the project until his untimely death in 1948. One of the greatest Swedish paleontologists Erik Jarvik took over the project and published a thorough anatomical studies of these two genera. See Figure 1, for a picture of the Ichthyostega taken from Erik Jarvik's tome "*Basic Structure and Evolution of Vertebrates, Vol 1.*"

Figure 1: Erik Jarvik's rendition of Ichthyostega.

Although no specimens preserved enough of the fingers or toes for an unambiguous count, Jarvik reconstructed Ichthyostega with *five* digits per limb. Why?

One needs to look at the history in order to understand such an unshakable faith in "pentadactylity" (five-fingered-ness). Richard Owen (one of England's greatest anatomists and a contemporary of Darwin) had actually hypothesized an archetypical tetrapod pentadactyl vertebrate. He was so pleased with it that he had a picture of this archetype engraved on his personal emblem. While Darwin held on to a more worldly view, he seemed to have been impressed by Owen's archetype which he described as a "*real representation as far as the most consummate skill and loftiest generalization can represent the parent form of the vertebrata.*"

In light of this history, it seems hardly surprising that Jarvik would accept the pentadactylity of vertebrates so easily, as many of my readers would. However, Jarvik did go a few steps ahead when he equated the "human culture" to the "basic pattern of our five-fingered hand."

However, the picture that Jarvik drew of an Ichthyostega came to some doubt beginning in 1984, when a soviet paleontologist O.A. Lebedev discovered a fossil of another early tetrapod *Tulerpeton* and finally in 1990, a joint Copenhagen-Cambridge team found a hind limb of Ichthyostega and a fore limb of Acanthostega. They wrote, reporting their disagreement with Jarvik's construction:

> "The proximal region of the hind limb of ichthyostega corresponds closely to the published description, but the tarsus [foot] and digits differ."

The back leg of Ichthyostega had *seven* toes.

Independently, quite a different line of investigation has emerged recently with our attempts to construct mechanized robot hands, capable of reproducing the same degree of dexterity as human hands. Notable among such hands are the Utah/MIT dextrous hand, the Stanford/JPL hand, the NYU Four Finger Manipulator (FFM), the Okada hand, and the Asada hand. Many early approaches had taken an anthropomorphic view and had justified the resulting kinematic structure with the belief that human hands represent some Platonic archetype structure. However, some rather beautiful geometric analysis of manipulative and grasping tasks have begun to seed doubts about the necessity of pentadactylity even in this mechanical domain. This essay studies these *other hands* and explores the underlying geometric and algorithmic ideas employed in this context.

To make matters somewhat concrete, we envisage an idealized dextrous hand, consisting of several independently movable force-sensing fingers. These fingers move as points in three-dimensional space. Here, we focus on the problem of *grip selection* for an object in the absence of static friction between the surface of the object and the fingers. This model is justified by the argument that presence of friction only improves the grasp and hence non-frictional grips represent in some sense the most pathological situation. Another argument favoring this grip model is that even when there is uncertainty about how much friction is available, the grasps synthesized under this restricted model remain immune to such uncertainties.

Such non-frictional grips have come to be known as *positive grips*. Since the fingers are assumed to be point fingers, a finger can only apply a force on the object along the surface-normal at the point of contact, directed inward.

If the shape of the object is precisely known then the problem of *grip selection* reduces to that of choosing a set of GRIP POINTS and a set of associated FORCE TARGETS. We then ask:

- *Can an arbitrary object be gripped (positively) with a finite number of fingers?*
- *If so, what are the grip points and the magnitudes of the forces exerted by the fingers (force targets) for such a grip?*

It can be then shown that almost every smooth object allows a positive grip with only a bounded (relatively small—but not five) number of fingers.

2 A Little Physics

A good starting place for us would be to understand how an object in equilibrium can be characterized. There are two ways of doing this: Either we can assume that the forces and torques acting on an object are *monogenic* (i.e., force/torques are derived from the potential energy as would be the case if the fingers are assumed to be compliant) or that the forces are *polygenic* (the force/torques applied at the fingers are generated by some actuators whose mechanics need not concern us). In the first case, we may proceed by looking at the local minima (stable equilibrium points) of the scalar function characterizing the energy. In the second case, we need to understand the *resultant* force and torque equation, as in the classical Newtonian mechanics. Here, our focus will be on the models corresponding to polygenic forces, as these models demonstrate the close connection between robotics and combinatorial geometry.

Consider a rigid body subject to a set of external polygenic forces f_1, ..., f_k, applied respectively at the points p_1, ..., p_k, as in Figure 2. Then the necessary and sufficient condition for the rigid body to be in equilibrium is that *the resultant force and the resultant torque must be null vectors*. In mathematical notations, this condition can be stated as follows:

$$\sum_{i=1}^{k} f_i = 0 \quad \text{and} \quad \sum_{i=1}^{k} p_i \times f_i = 0,$$

where the cross product $\tau = p \times f$ is defined as

$$\tau_x = p_y f_z - p_z f_y,$$

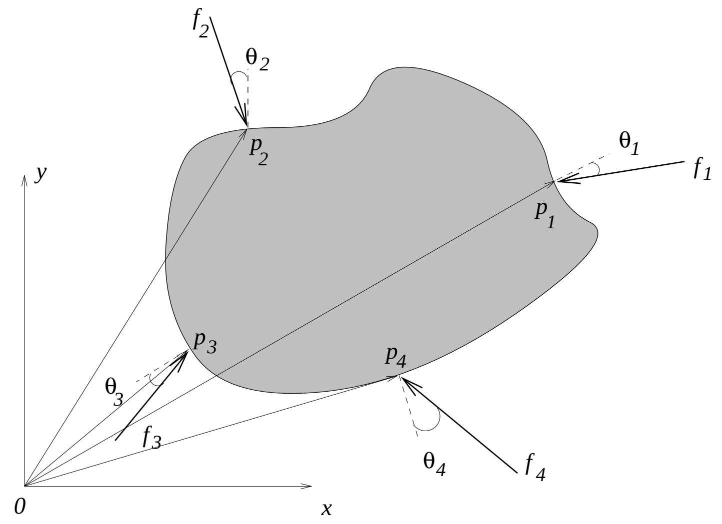

Figure 2: A planar object subject to four forces f_1, f_2, f_3 and f_4.

$$\begin{aligned} \tau_y &= p_z f_x - p_x f_z, \quad \text{and} \\ \tau_z &= p_x f_y - p_y f_x. \end{aligned}$$

Thus, in order to hold an object in equilibrium with a multi fingered hand (say, with k fingers), we need to place these fingers at points $p_1, \ldots, p_k$ on the boundary of the objects and apply forces $f_1, \ldots, f_k$ in such a manner that the equilibrium condition is satisfied. However, in this context, we need to satisfy two other conditions; namely,

- The normal components of the forces must be directed inward. Note that, otherwise, the fingers cannot maintain contact with the object.
- Furthermore, if the ith force f_i makes an angle θ_i with the surface normal at point p_i, then in order for the finger not to slip, $\tan\theta_i \le \mu$, where *coefficient of static friction* between the body and the finger is denoted by the constant μ.

Going back to the concept of *positive grip*, we realize that in this special case, we have no friction; the coefficient of static friction $\mu = 0$ and each θ_i must be zero. Thus if we write n_i for the unit normal to the surface of the body at the point p_i and directed inward, then the finger force f_i must be a nonnegative multiple of n_i. Thus

$$f_i = \alpha_i n_i, \quad \text{and} \quad p_i \times f_i = \alpha_i (p_i \times n_i), \quad \text{where } \alpha_i \ge 0, \text{ scalar.}$$

In order to better understand the effect of the requirements imposed by positive grip, we may consider the following somewhat easier problem:

Given: k grip points

$$\{p_1, p_2, \ldots, p_k\},$$

on the boundary of the body B.

Determine: If the object can be grasped (positively) by placing the fingers at the grip points.

For example, consider a planar rectangular object with four grip points at the mid points of the edges (shown in Figure 3.) Let the grip points be denoted as

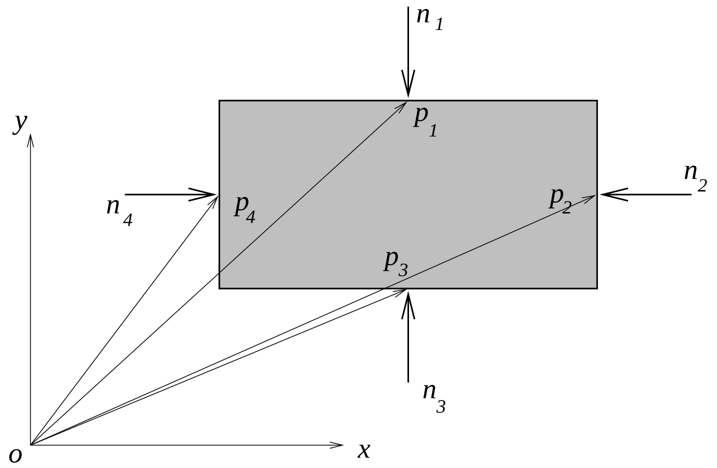

Figure 3: A planar rectangular object with designated grip points $\{p_1,\ p_2,\ p_3,\ p_4\}$.

p_1, p_2, p_3 and p_4 and the respective unit surface normals as n_1, n_2, n_3 and n_4. Then we wish to determine if there are four scalar quantities α_1, α_2, α_3 and α_4 such that

$$\begin{aligned}
\alpha_1 n_1 + \alpha_2 n_2 + \alpha_3 n_3 + \alpha_4 n_4 &= 0 \\
\alpha_1(p_1 \times n_1) + \alpha_2(p_2 \times n_2) + \alpha_3(p_3 \times n_3) + \alpha_4(p_4 \times n_4) &= 0 \\
\alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_3 \geq 0, \alpha_4 \geq 0 \quad \text{and} &\quad \text{not all } 0.
\end{aligned}$$

Note that, for this example, any choice of $\alpha_1 = \alpha_3$ and $\alpha_2 = \alpha_4$ will satisfy the conditions (assuming that at least two of them are nonzero and all of them are nonnegative). In particular, we could have chosen all the α's to be $1/4$!

To make matters little more abstract, we should define a *wrench map*, Γ, taking a point on the boundary of the object B to a point in the d-dimensional wrench space $\mathbb{R}^d$. Note that the term wrench space is used to denote a vector space consisting of all the wrenches. Its dimension d is 1, 3 or 6, depending on whether the object belongs to 1, 2 or 3-dimensional space.

$$\begin{aligned} \Gamma \,:\, \partial B &\rightarrow \mathbb{R}^d \\ :\, p_i &\mapsto (n_i, p_i \times n_i). \end{aligned}$$

Thus the wrench map Γ maps a point $p_i \in \partial B$ on the boundary of the body B to a wrench (a force/torque combination) that would be created if we apply a unit normal force directed inward at the point p_i. Then the feasibility of a positive grip can be expressed in terms of the existence of a solution of the following system of linear equations and inequalities:

$$\begin{aligned} \sum_{i=1}^{k} \alpha_i \Gamma(p_i) &= 0 \\ \alpha_i &\geq 0, i = 1, \ldots, k, \\ \sum_{i=1}^{k} \alpha_i &= 1. \end{aligned}$$

The last condition is added only for convenience, since if other α_i's were used, one could simply normalize their sum by dividing all the terms by a suitable denominator.

Geometrically, we are asking if some convex combination of the $\Gamma(p_i)$'s would yield the null vector. More compactly, we ask

$$0 \in \text{convex hull } (\Gamma(p_1), \ldots, \Gamma(p_k))?$$

If the answer to the preceding question is yes, then we can hold the object in equilibrium with the given grip points by applying forces whose magnitudes simply correspond to the coefficients used in the convex combination to express the null vector.

3 A Little Geometry

In this section, we shall provide some definitions, in order to discuss the geometry of wrench space in terms of the standard geometric vocabulary.

A d-dimensional space, $\mathbb{R}^d$, equipped with the standard linear operations, is said to be a *linear space*.

1. A *linear combination* of vectors p_1, ..., p_n from $\mathbb{R}^d$ is a vector of the form

   $$\alpha_1 p_1 + \cdots + \alpha_n p_n,$$

   where α_1, ..., α_n are in $\mathbb{R}$.

2. An *affine combination* of vectors $p_1, \ldots, p_n$ from $\mathbb{R}^d$ is a vector of the form

$$\alpha_1 p_1 + \cdots + \alpha_n p_n,$$

where $\alpha_1, \ldots, \alpha_n$ are in $\mathbb{R}$, with $\alpha_1 + \cdots + \alpha_n = 1$.

3. A *positive (linear) combination* of vectors $p_1, \ldots, p_n$ from $\mathbb{R}^d$ is a vector of the form

$$\alpha_1 p_1 + \cdots + \alpha_n p_n,$$

where $\alpha_1, \ldots, \alpha_n$ are in $\mathbb{R}_{\geq 0}$, the set of nonnegative real numbers.

4. A *convex combination* of vectors $p_1, \ldots, p_n$ from $\mathbb{R}^d$ is a vector of the form

$$\alpha_1 p_1 + \cdots + \alpha_n p_n,$$

where $\alpha_1, \ldots, \alpha_n$ are in $\mathbb{R}_{\geq 0}$ with $\alpha_1 + \cdots + \alpha_n = 1$.

By convention, we allow the empty linear combination (with $n = 0$) to take the value 0. We also assume that the empty linear combination is neither an affine combination nor a convex combination. Note that affine, positive and convex combinations are all linear combinations, and a convex combination is both affine and positive combinations.

A nonempty subset $L \subseteq \mathbb{R}^d$ is said to be a

1. *linear subspace*: if it is closed under linear combinations;
2. *affine subspace (or, flat)*: if it is closed under affine combinations;
3. *positive set (or, cone)*: if it is closed under positive combinations; and
4. *convex set*: if it is closed under convex combinations.

The intersection of any family of linear subspaces of $\mathbb{R}^d$ is again a linear subspace of $\mathbb{R}^d$. For any subset M of $\mathbb{R}^d$, the intersection of all linear subspaces containing M (i.e. the smallest linear subspace containing M) is called the *linear hull* of M (or, the linear subspace *spanned* by M), and is denoted by $\operatorname{lin} M$.

Similarly, the intersection of any family of affine subspaces, or positive sets or convex sets of $\mathbb{R}^d$ is again, respectively, an affine subspace or positive set or convex set. Thus for any subset M of $\mathbb{R}^d$, we can define

1. the *affine hull* (denoted by $\operatorname{aff} M$) to be the smallest affine subspace containing M,
2. the *positive hull* (denoted by $\operatorname{pos} M$) to be the smallest positive set containing M, and
3. the *convex hull* (denoted by $\operatorname{conv} M$) to be the smallest convex set containing M.

They are also called, respectively, the affine subspace, positive set and convex set *spanned* by M.

Equivalently, the linear hull lin M can be defined to be the set of all linear combinations of vectors from M. Similarly, the affine hull aff M (respectively, the positive hull pos M, the convex hull conv M) can be defined to be the set of all affine (respectively, positive, convex) combinations of vectors from M.

A set $p_1, \ldots, p_n$ of n vectors from $\mathbb{R}^d$ is said to be *linearly independent* if a linear combination

$$\alpha_1 p_1 + \cdots + \alpha_n p_n$$

can have the value 0, only when $\alpha_1 = \cdots = \alpha_n = 0$; otherwise, the set is said to be *linearly dependent*.

A set $p_1, \ldots, p_n$ of n vectors from $\mathbb{R}^d$ is said to be *affinely independent* if a linear combination

$$\alpha_1 p_1 + \cdots + \alpha_n p_n \quad \text{with } \alpha_1 + \cdots + \alpha_n = 0$$

can have the value 0, only when $\alpha_1 = \cdots = \alpha_n = 0$; otherwise, the set is said to be *affinely dependent*.

A *linear basis* of a linear subspace L of $\mathbb{R}^d$ is a set M of linearly independent vectors from L such that $L = \text{lin } M$. The dimension $\dim L$ of a linear subspace L is the cardinality of any of its linear basis.

An *affine basis* of an affine subspace A of $\mathbb{R}^d$ is a set M of affinely independent vectors from L such that $A = \text{aff } M$. The dimension $\dim A$ of an affine subspace A is one less than the cardinality of any of its affine basis.

Let C be any convex set. Then by *d-interior* of C, denoted $\text{int}_d\ C$, we mean the set of points p such that, for some d-dimensional affine subspace, A, p is interior to $C \cap A$ relative to A. If c is the dim aff C, then by an abuse of notation, we write int C to mean $\text{int}_c\ C$.

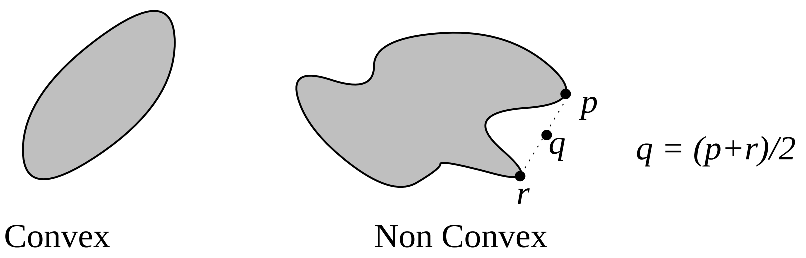

Figure 4: Convex and non convex planar sets.

Equivalently, we could have defined a convex set as follows:

> A subset of a linear space is *convex* if it contains with any two of its points the line segment defined by them. (See Figure 4.)

Examples of convex sets include: a point, a line segment, a simplex, a cone, a half space, an affine subspace or a linear subspace. Note that a closed half space can be defined to be the set of points

$$\{p = (x_1, \ldots, x_d) \in \mathbb{R}^d : a_1x_1 + \cdots + a_dx_d \geq b\}.$$

Thus if p and p' are two points in the closed half space then clearly every point on the line segment pp' also belongs to the closed half space. Note that since intersection of a family of convex sets is a convex set, we can also define convex hull of a subset $M \subseteq \mathbb{R}^d$ to be the *intersection of the family of all closed subsets containing* M.

Thus, given a finite set of points $M \subseteq \mathbb{R}^d$, we can enumerate the family of closed half spaces containing M and bounded by the affine hull of some subset of d points in M and then take their intersection to generate the convex hull of M. Such an algorithm would have a time complexity of $O(|M|^d)$. This naive algorithm can be improved significantly; techniques employed to construct convex hull efficiently occupy a central place in the nascent field of Computational Geometry [Ede87, O'Ro94].

3.1 Two Theorems from Convexity Theory

Following two theorems (Carathéodory's and Steintz's Theorems) from convexity theory have interesting implications to the theory of grasping. Subsequently, we will see how some of the most important results about grasping can be derived as simple consequences of these theorems. Proofs of these theorems can be found in any book on convexity theory [Eck93, Val64].

Carathéodory's Theorem: Let

$$X \subseteq \mathbb{R}^d \quad \text{and} \quad p \in \operatorname{conv} X.$$

Then there exists some subset $Y \subseteq X$ such that

$$|Y| \leq d+1 \quad \text{and} \quad p \in \operatorname{conv} Y. \quad \square$$

For example, if $d = 1$, then Carathéodory's theorem implies that a point in the convex hull of a set of points on a line belongs to a line segment defined by some two points from the set. Similarly, for $d = 2$, if $p \in \operatorname{conv} X$ then p belongs to the triangle formed by some three points of X (see Figure. 5), i.e.,

$$\exists \{y_1, y_2, y_3\} \subseteq X, p \in \operatorname{conv} \{y_1, y_2, y_3\}.$$

Steinitz's Theorem: Let

$$X \subseteq \mathbb{R}^d \quad \text{and} \quad p \in \operatorname{int} \operatorname{conv} X.$$

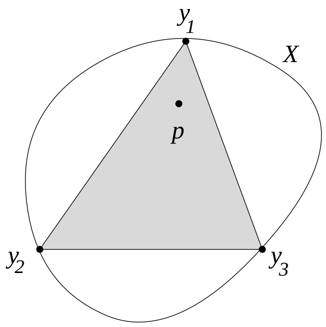

Figure 5: Example of Carathéodory's Theorem for $d = 2$.

Then there exists some subset $Y \subseteq X$ such that

$$|Y| \leq 2d \quad \text{and} \quad p \in \text{int conv}\, Y. \quad \square$$

Both Carathéodory's and Steinitz's theorems are examples of a general family of theorems, related to **Helly's Theorem**.

Helly's Theorem: Suppose $\mathcal{K}$ is a family of at least $d+1$ convex sets in $\mathbb{R}^d$, and $\mathcal{K}$ is finite or each member of $\mathcal{K}$ is compact. Then if each $d+1$ members have a common point, there is a point common to all members of $\mathcal{K}$. $\square$

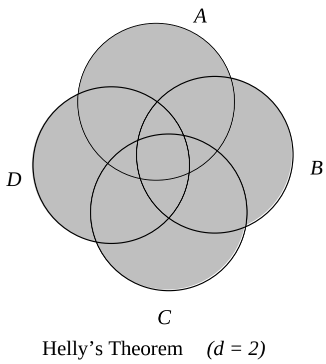

Figure 6: Example of Helly's Theorem for $d = 2$.

It is not hard to derive Carathéodory's theorem from Helly's theorem (using *polar duals*) and it can also be shown that Carathéodory's theorem implies Helly's theorem [Eck93]. Other members of the family of so-called Helly-type theorems include Radon's theorem, Tverberg's theorem, theorems of Kirchberger and Krasnosd'skiĭ, etc. Radon's theorem states that each set of $d+2$ or more points in $\mathbb{R}^d$ can be expressed as the union of two disjoint sets whose convex hulls have a common point.

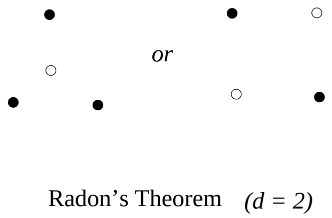

Figure 7: Example of Radon's Theorem for $d = 2$.

The proof of Carathéodory and Steinitz's theorems are not that involved. In order just to get a flavor of the techniques employed in combinatorial discrete geometry, we shall describe the proofs for the cases where $X \subseteq \mathbb{R}^2$ is a finite set of points in the plane.

Assume that $P = \operatorname{conv} X$ is a bounded polygon and p, a point of P. Let v $(\neq p)$ be a vertex of the polygon and let $\vec{vp}$ be the ray originating from the vertex v and passing through p. Let $\overline{vw} = P \cap \vec{vp}$ be the line segment with the end points v and $w \in \partial P$ on the boundary of P. There are two cases to consider: either w is a vertex of P or $w \in \overline{st}$ belongs to an edge $\overline{st}$ of P. In the first case, let $y_1 = v$ and $y_2 = w$ and in the later case, let $y_1 = v$, $y_2 = s$ and $y_3 = t$. By construction, $p \in \operatorname{conv}\{v, w\}$ and in the second case $w \in \operatorname{conv}\{s, t\} \Rightarrow p \in \operatorname{conv}\{v, s, t\}$. Now let $Y = \{y_1, y_2\}$ or $= \{y_1, y_2, y_3\}$, as the case may be. In either case, $Y \subseteq X$ and $|Y| \leq d + 1 = 3$. Similar constructive proofs for Carathéodory's theorem can be provided for higher dimension.

Now the proof of Steinitz's theorem can be given with a slight modification to the preceding argument. Let $P = \operatorname{conv} X$ be as before and p, a point in the interior of P. Now construct a ray $\vec{vp}$ as before, except that v is now chosen to be an interior point on an edge $\overline{ab}$ of P. Let $\overline{vw} = P \cap \vec{vp}$ be the line segment as before with w on the boundary of P. Again there are two cases to consider: either w is a vertex of P or $w \in \overline{st}$ belongs to interior of an edge $\overline{st}$ of P. In the first case, let $y_1 = a$, $y_2 = b$ and $y_3 = w$ and in the later case, let $y_1 = a$, $y_2 = b$, $y_3 = s$ and $y_4 = t$. Now let $Y = \{y_1, y_2, y_3\}$ or $= \{y_1, y_2, y_3, y_4\}$, as the case may be. In either case, $Y \subseteq X$ and $|Y| \leq 2d = 4$ and $p \in \operatorname{int} \operatorname{conv} Y$. The extension to higher dimension is also fairly straightforward. The fact that the bound $2d$ is tight for Steinitz's theorem can be seen from the two dimensional example shown in Figure 8.

4 A Little Robotics

Equipped with our understanding of the geometric structures in convexity theory, we are now ready to tackle one of the simplest (but rather interesting) problems

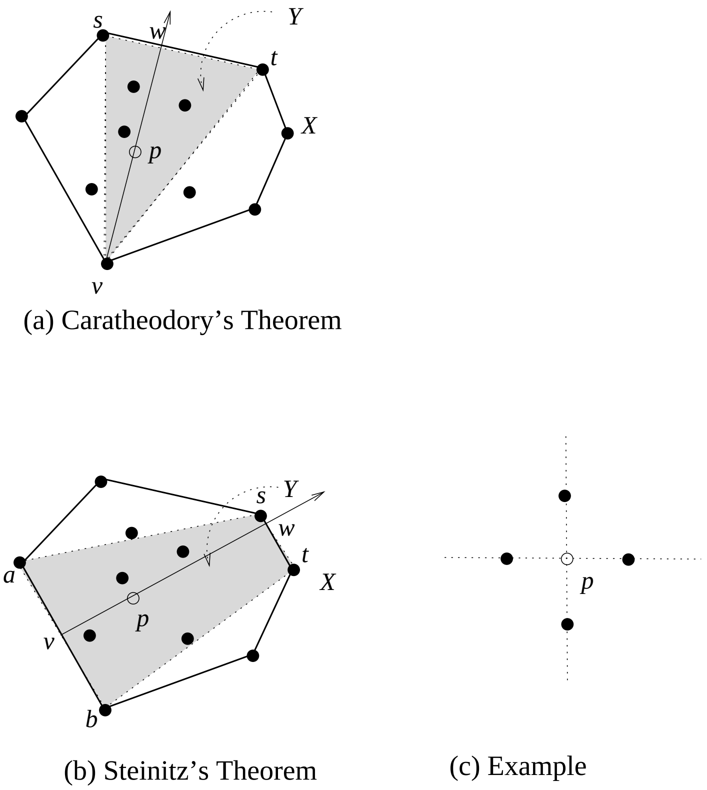

Figure 8: Proof sketches.

in grasping theory:

Existence of Positive Grips:

Given: An arbitrary rigid 3-dimensional object B and some number k.

Determine: Whether one can choose k (finite) grip points, $\{p_1, p_2, \ldots, p_k\} \subseteq \partial B$ on the boundary of B such that the object can be grasped (positively) by placing fingers at those grip points.

$$\left(\exists ? \{p_1, \ldots, p_k\} \subseteq \partial B\right) \left[0 \in \operatorname{conv}\left(\Gamma(p_1), \ldots, \Gamma(p_k)\right)\right].$$

Surprisingly, the answer to the problem turns out to be "yes" and the necessary number of fingers is 7. That is, there is a "universal hand" with seven fingers that can grasp *any* rigid object by judiciously choosing the grip points. Of course when we say *any*, we actually make some reasonable assumptions about the object. Namely, we assume that *B is a closed bounded connected object with piece-wise smooth boundary ∂B.*

The proof proceeds in three simple steps:

STEP 1: Show that

$$0 \in \text{conv } \Gamma(\partial B),$$

where $\Gamma \colon \partial B \to \mathbb{R}^6 : p \mapsto (n, p \times n)$. This is a simple consequence of the fact that an object under uniform pressure remains in equilibrium. The proof of this claim can be given rigorously using the *Divergence theorem of Gauss*.

STEP 2: By Carathéodory's theorem

$$\Big(\exists \, \{\Gamma(p_1), \ldots, \Gamma(p_k)\} \subseteq \Gamma(\partial B)\Big) \, \Big[k \leq 7 \text{ and } 0 \in \text{conv}\,(\Gamma(p_1), \ldots, \Gamma(p_k))\Big].$$

Hence there are positive nonnegative scalar quantities α_1, ..., α_k such that:

$$\begin{aligned} \alpha_1 n_1 + \cdots + \alpha_k n_k &= 0, \\ \alpha_1(p_1 \times n_1) + \cdots + \alpha_k(p_k \times n_k) &= 0. \end{aligned}$$

STEP 3: The positive grip is then selected by choosing as grip points

$$\begin{aligned} \text{Grip Points} &= \{p_1, \ldots, p_k\} \subseteq \partial B, \\ \text{Force Magnitudes} &= \alpha_1, \ldots, \alpha_k, \end{aligned}$$

with k no larger than 7.

4.1 Problems with Equilibrium Grasps

Similar arguments in the plane implies that *any*[1] planar object can be grasped by at most *four* fingers. The number four is arrived at by taking the dimension of the wrench space and adding one to it, as implied by the Carathéodory's theorem. It is also instructive to examine a set of equilibrium grasps for three planar objects: a rectangle, a triangle and a disc. First consider the grasps for the rectangle. Clearly, the grasps (a) and (d) are not as secure as (g)—a horizontal external force will break the grasp (a) and an external torque about the center of the rectangle will break the grasp (d). In comparison, the grasp (g) is immune to such external disturbances, provided of course that such disturbances are relatively small in magnitude. Similar examination will show that the grasp (h) is the most secure for a triangle. However, in the case of the disc, while the grasps (f) and (i) are better than (c), there is simply no way to resist an external torque about the center irrespective of how many fingers are used.

[1] Closed, connected and bounded with piece wise smooth boundary.

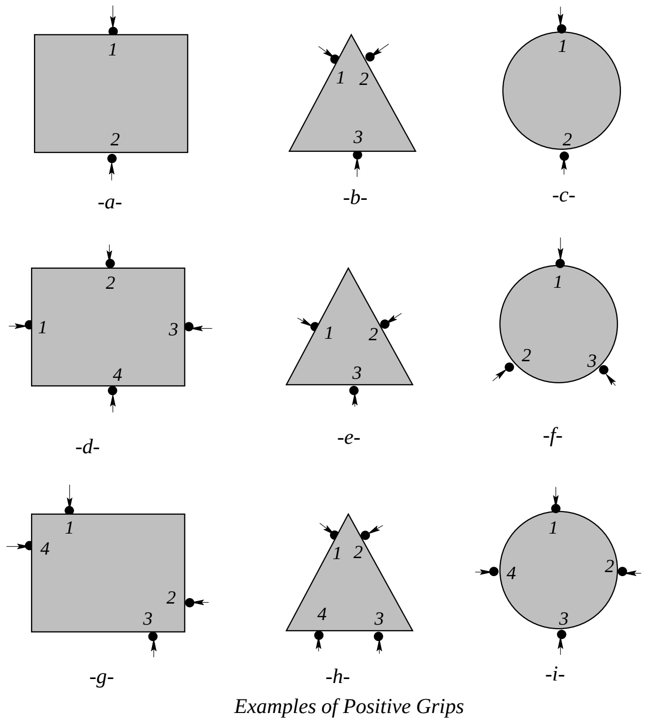

Figure 9: Grasping planar objects.

The kinds of secure grasps described in the preceding paragraph have been characterized as *closure grasps*. Furthermore, exactly those objects that do not allow closure grasps can also be characterized in purely geometric terms, and are referred to as *exceptional objects*. While we shall not go into a detailed description of the exceptional objects (see [MSS87]), it should suffice for the present purpose to say that the only planar bounded exceptional object is a disc and the only spatial bounded exceptional object is an object bounded by a surface of revolution[2].

Closure Grasps: A set of grip points on an object B is said to constitute a *closure grasp* if and only if any arbitrary external force/torque combination acting on the object can be balanced by simply *pressing the fingertips against this object at the selected fixed grip points*.

[2]If one allows unbounded objects then in 3-D, we have to include unbounded prisms and helical objects and in 2-D an unbounded strip of constant width. These objects in 3-D describe the so-called Reuleux pairs, studied almost a century ago.

Thus our job is to

Given: An arbitrary non-exceptional rigid 3-dimensional object B and some number k.

Determine: If one can choose k (finite) grip points, $\{p_1, p_2, \ldots, p_k\} \subseteq \partial B$ on the boundary of B such that the object can be grasped (with closure) by placing fingers at those grip points.

In other words, we must have for every $g \in \mathbb{R}^6$ ($-g$ is the external wrench), a set of nonnegative force magnitudes

$$\begin{array}{rcl} \{\alpha_1, \ldots, \alpha_k\} & & \alpha_i \geq 0 \\ \alpha_1 \Gamma(p_1) + \cdots + \alpha_k \Gamma(p_k)) & = & g \end{array}$$

Equivalently,

$$\mathrm{pos}\,(\Gamma(p_1), \ldots, \Gamma(p_k)) = \mathbb{R}^6;$$

$\Gamma(p_1), \ldots, \Gamma(p_k)$ positively span the entire wrench space. Since this is also equivalent to the following condition

$$\left(\exists \epsilon > 0\ \forall g \in \mathbb{R}^6\right) \left[\epsilon g = \sum \alpha_i \Gamma(p_i), \sum \alpha_i = 1, \alpha_i \geq 0\right],$$

we can also express this problem as asking

$$\left(\exists ? \{p_1, \ldots, p_k\} \subseteq \partial B\right) \left[0 \in \mathrm{int\ conv}\,(\Gamma(p_1), \ldots, \Gamma(p_k))\right].$$

Note that the first condition implies that a sufficiently small 6-dimensional ball (of radius ϵ) can fit in the convex hull of the wrenches and hence leads to the second condition.

The answer to the problem turns out to be "yes" (for all non-exceptional objects) and the necessary number of fingers is 12 (twice the dimension of the wrench space).

The proof proceeds again in three simple steps:

STEP 1: Show that, if B is non-exceptional

$$0 \in \mathrm{int\ conv}\ \Gamma(\partial B),$$

where $\Gamma \colon \partial B \to \mathbb{R}^6 : p \mapsto (n, p \times n)$. This is only true if B is non-exceptional, as otherwise $\Gamma(\partial B)$ spans only a low-dimensional subspace.

STEP 2: By Steinitz's theorem

$$\left(\exists\ \{\Gamma(p_1), \ldots, \Gamma(p_k)\} \subseteq \Gamma(\partial B)\right) \left[k \leq 12 \text{ and } 0 \in \mathrm{int\ conv}\,(\Gamma(p_1), \ldots, \Gamma(p_k))\right].$$

STEP 3: The closure grasp is then selected by choosing as grip points

$$\text{Grip Points } = \{p_1, \ldots, p_k\} \subseteq \partial B,$$

with k no larger than 12.

Some of the results about grasping (including the ones discussed earlier) has been summarized in the following table:

	2D Objects	3D Objects
Equilibrium Grasps		
Piecewise Smooth	4	7
Smooth	3	5
Convex, Smooth	2	2
Closure Grasps		
Piecewise Smooth	6 (excluding disks)	12 (excluding objects with a surface of revolution)

Table 1: Summary of Results. The numbers in the table are upper bounds on the required number of fingers.

5 A Simple Algorithm

At this point, it is natural for a roboticist to ask how one (a robot) can construct a grasp for a specific object and what sorts of computation this may entail. The answer turns out to be very interesting and shows a close connection of this problem to a classical algorithm, "*the simplex method*," used for solving linear programming problems.

Thus, suppose we have a polyhedral object with n faces. Since this object is "non-exceptional," in principle, we should be able to grasp it by a closure grasp using no more than twelve fingers. In our terminology, we wish to simply identify no more than twelve grip points on the faces of the polyhedron—but we wish to do so *constructively*, and furthermore, as quickly as possible.

We proceed in a manner not very dissimilar from the ways we proved the existences of such a grasp. We first create a closure grasp with extremely large number of fingers: about $15n$ grip points, where n is the number of faces of the polyhedron. Of course, this is all done by our imagination (or by a mathematical model in the computer memory); we don't need to physically construct a hand with $15n$ fingers! Next, step by step, we can eliminate one finger in each step while maintaining closure grasp as long as the number of grip points at the beginning of that step is strictly larger than twelve. The algorithm terminates when we are left with no more than twelve grip points.

5.1 Algorithmic Preliminaries

In order to understand the process by which the fingers are eliminated, we shall digress to consider an algorithmic approach to *algebraic manipulation with positive linear combinations*.

Given: A set of vectors $\{V_1, V_2, \ldots, V_l\} \subseteq \mathbb{R}^d$ and $V \in \mathbb{R}^d$ such that

$$\begin{aligned} \alpha_1 V_1 + \cdots + \alpha_l V_l &= \alpha V \\ & \alpha_i \geq 0, \alpha > 0, V \neq 0. \end{aligned}$$

Find: A subset $m \leq d$ vectors

$$\{V_{i_1}, V_{i_2}, \ldots, V_{i_m}\} \subseteq \{V_1, \ldots V_l\} \quad \text{and} \quad \alpha' > 0$$

such that

$$\begin{aligned} \alpha'_1 V_{i_1} + \cdots + \alpha'_m V_{i_m} &= \alpha' V \\ & \alpha'_i \geq 0, (\alpha' > 0, V \neq 0). \end{aligned}$$

The problem can be solved by the algorithm described below. If you are already familiar with the simplex algorithm for linear programming problems, then you should realize that the basic step of the algorithm shown resembles the "pivot step" of the simplex algorithm.

Reduction Algorithm

if $l \leq d$ then HALT;

else repeat

Choose d vectors from $\{V_1, \ldots, V_l\}$
(Say, the first d): $\{V_1, \ldots, V_d\}$
There are two cases to consider, depending on whether the vectors $V_1, \ldots, V_d$ are *linearly dependent* or not.

Case 1: $V_1, \ldots, V_d$ **are linearly dependent.**
We can write

$$\beta_1 V_1 + \cdots + \beta_d V_d = 0,$$

not all $\beta_i = 0$.
Assume that at least one $\beta_i < 0$ (otherwise, replace each β_i by $-\beta_i$ in the equation to satisfy the condition.)

Let
$$\gamma = \min_{\beta_i < 0}(\alpha_i / \beta_i) < 0.$$
(For specificity, we may assume $\gamma = \alpha_1/\beta_1$.)
Put $\alpha'_i = \alpha_i - \gamma\beta_i$ for $1 \le i \le d$.
Hence by adding the equation $(\sum_{i=1}^{l} \alpha_i V_i = \alpha V)$ to $(-\gamma \sum_{i=1}^{d} \beta_i V_i = 0)$, we get

$$\alpha'_2 V_2 + \cdots + \alpha'_d V_d + \alpha_{d+1} V_{d+1} + \cdots + \alpha_l V_l = \alpha V,$$

and by construction $\alpha'_2, \ldots, \alpha'_d \ge 0$.

Case 2: $V_1, \ldots, V_d$ **are linearly independent.**
We can write
$$\beta_1 V_1 + \cdots + \beta_d V_d = V.$$
Assume that at least one $\beta_i < 0$ (otherwise, we have nothing more to do!)
Let
$$\gamma = \min_{\beta_i < 0}(\alpha_i / \beta_i) < 0.$$
(For specificity, we may assume $\gamma = \alpha_1/\beta_1$.)
Put $\alpha'_i = \alpha_i - \gamma\beta_i$ for $1 \le i \le d$, and $\alpha' = \alpha - \gamma > 0$.
Hence by adding the equation $(\sum_{i=1}^{l} \alpha_i V_i = \alpha V)$ to $(-\gamma \sum_{i=1}^{d} \beta_i V_i = -\gamma V)$, we get

$$\alpha'_2 V_2 + \cdots + \alpha'_d V_d + \alpha_{d+1} V_{d+1} + \cdots + \alpha_l V_l = \alpha' V,$$

and by construction $\alpha'_2, \ldots, \alpha'_d \ge 0$.

Note that this process terminates after at most $(l - d)$ repetitions of the basic step and each basic step involves some matrix operations involving $d \times d$ matrices, thus using in each basic step amount of computer time that is cubic in d. [In algorithmic terminology, we would write that "the reduction algorithm has a time complexity of $O(ld^3)$."] In our grasping application, d will turn out to be a constant $(= 6)$ and l no more than $15n$. Thus we will see that this algorithm will give us a grasping algorithm whose time complexity will be proportional to n, the number of faces of the polyhedron it is trying to grasp.

—The end of digression.

5.2 Grasping Algorithms

Let us get back to our original question about grasping a polyhedron B with n faces. As hinted earlier, we shall start with a closure grasp of B using no more than $15n$ grip points. Assume that B is provided with a triangulation of each face, and

$$t_1, t_2, \ldots, t_N$$

is the set of triangles partitioning ∂B. For each triangle t_i, choose three non-collinear grip points p_{i_1}, p_{i_2} and $p_{i_3} \in t_i$ such that $(p_{i_1} + p_{i_2} + p_{i_3})/3$ is the centroid of t_i. In totality they will give us the initial $3N$ grip points. Using Euler's formula and some simple combinatorics, one can show that $N \leq 5n - 12$ and the total number of grip points is no more than $15n - 36$ (see [MSS87]).

Now, it can be shown that if one chooses p_{i_j}'s, $1 \leq i \leq N$, $j = 1, 2, 3$, as the grip points then they give rise to a closure grasp. In particular, we can see [MSS87] (by using linear algebraic manipulations) that

$$\begin{aligned} &\frac{\text{Area}\,(t_1)}{3}\Gamma(p_{1_1}) + \frac{\text{Area}\,(t_1)}{3}\Gamma(p_{1_2}) + \frac{\text{Area}\,(t_1)}{3}\Gamma(p_{1_3}) \\ &\quad + \cdots + \frac{\text{Area}\,(t_N)}{3}\Gamma(p_{N_1}) + \frac{\text{Area}\,(t_N)}{3}\Gamma(p_{N_2}) + \frac{\text{Area}\,(t_N)}{3}\Gamma(p_{N_3}) = 0, \end{aligned}$$

and that

$$\text{pos}\Big(\Gamma(p_{1_1}), \Gamma(p_{1_2}), \ldots, \Gamma(p_{N_3})\Big) = \mathbb{R}^6.$$

Henceforth, rewriting these grip points as $\{p_1, p_2, \ldots, p_l\}$, and the "area terms" as magnitude of coefficients: $\alpha_1, \alpha_2, \ldots, \alpha_l$, we have

$$\alpha_1 \Gamma(p_1) + \alpha_2 \Gamma(p_2) + \cdots + \alpha_l \Gamma(p_l) = 0, \tag{1}$$

where $\alpha_i > 0$. Furthermore, since

$$\text{lin}\Big(\Gamma(p_1), \Gamma(p_2), \ldots, \Gamma(p_l)\Big) = \mathbb{R}^6,$$

without loss of generality, assume that the first six wrenches are linearly independent, thus spanning the entire wrench space, i.e.,

$$\text{lin}\Big(\Gamma(p_1), \ldots, \Gamma(p_6)\Big) = \mathbb{R}^6.$$

Synthesizing a Equilibrium Grasp with Seven Fingers Let us now see how we can go from here to get a simple equilibrium grasp with no more than seven fingers. Note first that we can rewrite our equation 1 (for l-fingered grip) as

$$\frac{\alpha_1}{\alpha_l}\Gamma(p_1) + \cdots + \frac{\alpha_{l-1}}{\alpha_l}\Gamma(p_{l-1}) = -\Gamma(p_l),$$

where $\alpha_i > 0$ and $\Gamma(p_i) \in \mathbb{R}^6$. Now, we can use the "Reduction Algorithm" to find

$$\{p_{i_1}, p_{i_2}, \ldots, p_{i_m}\} \subseteq \{p_1, \ldots, p_{l-1}\}$$

satisfying the conditions below:

$$\alpha'_1\Gamma(p_{i_1}) + \cdots + \alpha'_m\Gamma(p_{i_m}) = -\alpha'\Gamma(p_l),$$

and $m \leq 6$. Thus we have

$$\alpha'_1\Gamma(p_{i_1}) + \cdots + \alpha'_m\Gamma(p_{i_m}) + \alpha'\Gamma(p_l) = 0,$$

with $\alpha'_1 \geq 0$, ..., $\alpha'_m \geq 0$ and $\alpha' > 0$. Of course, this is our equilibrium grasp using no more than $m+1 \leq 7$ fingers, placed at grip points p_{i_1}, ..., p_{i_m}, p_l with associated force magnitudes α'_1, ..., α'_m, α'.

Synthesizing a Closure Grasp with Twelve Fingers Recall that the initial l grip points are so chosen that

$$\text{lin}\left(\Gamma(p_1), \ldots, \Gamma(p_6)\right) = \mathbb{R}^6.$$

Let

$$V = -(\Gamma(p_1) + \ldots + \Gamma(p_6)).$$

Express V using all the wrenches as follows

$$\alpha_1\Gamma(p_1) + \alpha_2\Gamma(p_2) + \cdots + \alpha_l\Gamma(p_l) = V,$$

which exploits the fact that the original set of l grip points form a closure grasp (i.e., $\Gamma(p_i)$'s positively span the entire wrench space).

Now, we can again use the "Reduction Algorithm" to find

$$\{p_{i_1}, p_{i_2}, \ldots, p_{i_m}\} \subseteq \{p_1, \ldots, p_{l-1}\}$$

satisfying the conditions below:

$$\alpha'_1\Gamma(p_{i_1}) + \cdots + \alpha'_m\Gamma(p_{i_m}) = V,$$

and $m \leq 6$. We now choose as the desired grip points

$$\{p_{i_1}, p_{i_2}, \ldots, p_{i_m}\} \cup \{p_1, \ldots, p_6\},$$

numbering no more than $m + 6 \leq 12$. We claim that these give rise to a closure grasp.

To see why, consider some arbitrary external wrench $f \in \mathbb{R}^6$. We wish to show that this f can be expressed as a positive linear combination of

$$\{\Gamma(p_{i_1}), \ldots, \Gamma(p_{i_m})\} \cup \{\Gamma(p_1), \ldots, \Gamma(p_6)\}.$$

First, note that we can write f as a linear combination of $\Gamma(p_i)$, $i = 1$, ..., 6:

$$f = \sum_{i=1}^{6} \beta_i\Gamma(p_i),$$

and suppose that not all $\beta_i \geq 0$, since otherwise we have nothing more to prove. Now let

$$\gamma = \min_{1 \leq i \leq 6} \beta_i < 0.$$

Thus

$$\begin{aligned}
f &= \sum_{i=1}^{6} \beta_i \Gamma(p_i) \\
&= \sum_{i=1}^{6} (\beta_i - \gamma)\Gamma(p_i) + (-\gamma) \sum_{i=1}^{6} -\Gamma(p_i) \\
&= \sum_{i=1}^{6} (\beta_i - \gamma)\Gamma(p_i) + (-\gamma)V \\
&= \sum_{i=1}^{6} (\beta_i - \gamma)\Gamma(p_i) + \sum_{j=1}^{m} (-\gamma\alpha'_j)\Gamma(p_{i_j}).
\end{aligned}$$

Since $-\gamma$ is positive and $\gamma < \beta_i$ and since α'_j's are positive, all the coefficients in the above equation are nonnegative.

Thus, we have

$$\mathrm{pos}\left(\{\Gamma(p_{i_1}), \ldots, \Gamma(p_{i_m})\} \cup \{\Gamma(p_1), \ldots, \Gamma(p_6)\}\right) = \mathbb{R}^6,$$

which means we have shown that the chosen grip points $\{p_{i_1}, \ldots, p_{i_m}\} \cup \{p_1, \ldots, p_6\}$ indeed form a closure grasp.

6 Final Remarks

Most of the questions dealt here come from one of the first papers I wrote in this area about ten years ago with Jack Schwartz and Micha Sharir. Since then the area has grown substantially and researchers have addressed many more interesting questions dealing with different finger models, different concepts of closure, various measures of goodness of a grasp, regrasping (also called finger-gaiting), fixturing and workholding. Of course, it is not possible to go into all these topics here. The readers wishing to learn more about these topics must consult the references given at the end of the paper. Also, a web-site designed by Ken Goldberg (FixtureNet, URL `http://teamster.usc.edu/fixture/`) at University of Southern California can automatically find for you how a polygonal (2D) object can be fixtured. You may want to check it out to sharpen your intuition.

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References

[DGK63] L. Danzer, B. Grünbaum and V. Klee. "Helly's Theorem and its Relatives," *Convexity*, Proc. of Symposia in Pure Math., Vol. **7**, pp. 101–180, AMS, Providence, RI, 1963.

[Eck93] J. Eckhoff. "Helly, Radon, Carathéodory Type Theorems." *Handbook of Convex Geometry*, Vol. **A**, (Ed. P.M. Gruber and J.M. Willis), pp. 389–448, North-Holland, New York, 1993.

[Ede87] H. Edelsbrunner. *Algorithms in Combinatorial Geometry*, Springer-Verlag, New York, 1987.

[Gou93] S.J. Gould. *Eight Little Piggies*, Chapter 4, pages 63–78. W.W. Norton & Company, New York. 1993.

[HLM+90] J. Hong, G. Lafferriere, B. Mishra, and X. Tan. "Fine Manipulation with Multifinger Hands." In *1990 IEEE International Conference on Robotics and Automation*, pp. 1568–1573, May 1990.

[KMY92] D. Kirkpatrick, B. Mishra, and C. Yap. "Quantitative Steinitz's Theorem with Applications to Multifingered Grasping." *Discrete & Computational Geometry*, Springer-Verlag, New York, **7**(3):295–318, 1992.

[Lak78] K. Lakshminarayana. "The Mechanics of Form Closure." *ASME 78-DET-32*, 1978.

[MJKS85] M.T. Mason and J.K. Salisbury, Jr. *Robot Hands and the Mechanics of Manipulation*. MIT Press, 1985.

[MNP90] X. Markenscoff, L. Ni, and C.H. Papadimitriou. "The Geometry of Grasping." *The International Journal of Robotics Research*, **9**(1), 1990.

[MS89] B. Mishra and N. Silver. "Some Discussion of Static Gripping and Its Stability." *IEEE Transactions on Systems, Man and Cybernetics*, **19**:783–796, 1989.

[MSS87] B. Mishra, J.T. Schwartz, and M. Sharir. "On the Existence and Synthesis of Multifinger Positive Grips." *Algorithmica*, **2**:541–558, 1987.

[O'Ro94] J. O'Rourke. *Computational Geometry in C*, Cambridge University Press, Cambridge, 1994.

[Ste] E. Steinitz. "Bedingt Konvergente Reihen und Konvexe Systeme." *J. reine angew. Math.*, (I) **143**:128–175, 1913; (II) **144**:1–48, 1914; and (III) **146**:1–52, 1916.

[SY87] J.T. Schwartz and C.-K. Yap, editors. *Advances in Robotics, Vol. I: Algorithmic and Geometric Aspect of Robotics*. Lawrence Erlbaum Associates, Publishers, Hillsdale, New Jersey, 1987.

[Tei95] M. Teichmann. *Grasping and Fixturing: a Geometric Study and an Implementation*. Ph.D. Thesis, New York University, New York, 1995.

[Val64] F.A. Valentine. *Convex Sets*. McGraw-Hill, New York, 1964.