A Survey of

Computational Differential Algebra

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Abstract

In this note, we explore the computational aspects of several problems in differential algebra with concrete applications in dynamics and motion-planning problems in robotics, automatic synthesis of control schemes for nonlinear systems and simulation of physical systems with fixed degrees of freedom.

Our primary aim is to study, compute and structurally describe the solution of a system of differential equations with coefficients in a field (say, the field of complex numbers, $\mathbb{C}$). There seem to have been various approaches in this direction: e.g. ideal theoretic approach of Ritt, Galois theoretic approach of Kolchin and Singer and group theoretic technique of Lie. It is interesting to study their interrelationship and effectivity of various problems they suggest.

In general, these problems are known to be uncomputable; thus, we need to understand under what situations these problems become feasible. As related computer science questions, we also need to study the complexity of these problems, underlying data-structures, effects of the representation (e.g. sparsity).

Of related interest are some closely-related problems such as symbolic integration problem, solving difference equations, integro-differential equations and differential equations with algebraic constraints.

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1 General Introduction

The subject of this survey is the emerging field of *computational differential algebra and geometry.* In a manner similar to *computational combinatorial geometry* and *computational algebraic geometry*, this emerging field addresses the computational, algorithmic and complexity questions that arise naturally as one studies concrete and applicable versions of constructive differential algebra and differential geometry. Clearly, it has many historical and technical connections with *computational algebra* and forms a significant portion of any practical *symbolic computation package*. However, there are so many unanswered and unexplored foundational problems in this area, that in spite of its clear-cut applicability and unlike its sister branches, the field appears to be still in its early infancy.

Let us begin by looking at the problem of integration. (See [20, 43].) That is given a univariate *elementary function* as an integrand, we would like to compute its *indefinite integral*. For the present, we may simply think of the *class of elementary functions* as the class consisting of algebraic functions, logarithms and exponentials with *arbitrary levels* of self and mutual nesting. For instance, this class includes such functions as:

$$\begin{array}{llll} x, & \log x, & \log\log x, & \ldots \\ & \exp x, & \exp\exp x, & \ldots \\ & \sin x = \frac{1}{2i}(\exp(ix) - \exp(-ix)), & \sinh x = \frac{1}{2}(\exp(x) - \exp(-x)), & \\ & \mathrm{arc}\ \tan x = \frac{1}{2i}\log\left(\dfrac{1+ix}{1-ix}\right), & x^{\sqrt{2}} = \exp(\sqrt{2}\log x), & \end{array}$$

Thus, our problem is to compute the indefinite integral g of an elementary function f as follows: if an elementary g exists, output that g, or else report failure.

$$g(x) = \int f(x)\, dx.$$

In a simpler term, we want to determine if the following differential equation has an elementary solution

$$\frac{d}{dx} g - f(x) = 0,$$

assuming that f is elementary.

Note that not all elementary functions have elementary indefinite integrals. For instance, if $f(x) = \exp(-x^2)$ then $\int f(x)dx = \sqrt{\pi/2}\,\mathrm{erf}\,(x)$, which is not elementary. Yet another example is $\int dx/\log x$. However, in the case when f *does* have an elementary indefinite integral, its form is known by a beautiful classical theorem of Liouville and there are several algorithms (due to Davenport, Hermite, Horowitz, Risch, Rothstein, Trager) that can effectively compute the solution.

Thus, it is reasonable to ask whether it is possible to decide *if a given elementary function has an elementary indefinite integral*. However, the answer is not at all clear-cut. Let's assume that our underlying field of constants is the rational numbers adjoined with one algebraic number $i = \sqrt{-1}$ and one transcendental number π; that is, we are working with $\mathbb{Q}(i, \pi)$. Now, consider the following integral:

$$\int \frac{\log \exp x - x}{2\pi i} \exp(-x^2) dx.$$

If we simplify $\log \exp x = x$ then the answer is a constant (say 1), which is elementary. But if we simplify $\log \exp x = x + 2\pi i$ then the answer is $\sqrt{\pi/2}\,\mathrm{erf}\,(x)$, which is not elementary. We can create more complicated examples.

Let's say w_1, w_2, …, w_9 are new symbols and

$$P(w_1, w_2, \ldots, w_9) \in \mathbb{Z}[w_1, w_2, \ldots, w_9],$$

is a polynomial in these nine variables. Now let $\widehat{P}(x)$ be obtained from P by replacing each w_j by the following expression:

$$\frac{\log \exp x^j - x^j}{2\pi i}.$$

Thus we see that there is a way to simplify $\widehat{P}(x)$ to 0 if and only if $P(w_1, w_2, \ldots, w_9)$ has an integral solution in $\mathbb{Z}$. This is the classical Diophantine problem, and if P is the universal Diophantine equation [10, 26](which can be expressed with no more than 9 variable) then there is no effective general process to determine if this equation is solvable. Now asking if the following integration problem

$$\int \widehat{P}(x) \exp(-x^2) dx$$

has an elementary solution is simply reiterating the classical Diophantine problem in a different guise.

Of course, one may reasonably argue that the obstacle to integrating as presented here has very little to do with the "integration problem;" rather, it's the result of the multivalued nature of the underlying elementary functions. This has been the standpoint of Risch and many others and has been the starting point for an eminently successful research effort in this field.

The moral of the story is simple: it is not uncommon to find many important questions in this field to be undecidable or of unresolved status; however, this precarious situations should be hardly discouraging. Much progress can be accomplished by posing the questions in properly useful settings. In particular, computational complexity theoretic views, accompanied by a healthy amount of practical experimentations, could play a significant role in improving our understanding of this field.

Also, the techniques of computational algebra (e.g. Gröbner bases, characteristic sets, resultants) may help the field considerably. It should be noted that there is already a considerable amount of exchange of ideas: the techniques of Ritt in differential algebra have been successfully used in geometric theorem proving; the ideas of H-bases and G-bases have been generalized to a limited extent for applications in differential algebra; Kovacic's Galois theoretic techniques for differential equations have found applications in certain implementations of difference equation solvers.

This document is organized as follows: The starting point for us will be Ritt-Kolchin-Kaplansky's differential algebra. Here, we introduce the general concepts and some thorny unsolved questions of fundamental interest to the field. Next, we discuss some of the application areas arising from the relation with control theory, robotics and dynamical systems.

2 Differential Algebra and Some Open Problems

Currently, there is much interest in differential rings, originally introduced by J.F. Ritt, as well as in extending some key ideas of computational algebraic ring theory to their differential analogs. (See for example [5], [6], [17], [38], [39], [40] and [41]). The reasons for this interest are both practical and theoretical. There is, in fact, a growing effort to use differential algebra in order to solve problems in control theory, dynamical systems and robotics. From the theoretical point of view, it is equally important that we understand the precise relation between 'old' constructive methods (Ritt-Seidenberg algorithm [46]) and the recent Gröbner bases-like approach.

A constructive study of differential algebra also promises to give new insight into its quite complicated structures. For example, rings of differential polynomials are not Noetherian, hence differential ideals can be much more complex than algebraic ideals. The discussion in [17] shows that one can construct examples of differential ideals that are not even recursive!

On the other hand the structure of differential ideals is not completely unruly, and one can hope to characterize classes of differential rings and of ideals for which suitable algorithmic techniques can be developed. The concept of H-bases for differential ideals is one such important ideas that have begun to be studied quite extensively.

The differential algebras considered here are commutative rings[1] of differential polynomials in several differential indeterminates over a field of constants. (See the classical works of Janet [25], Kaplansky [27], Kolchin [29], Riquier [42] and Ritt [45].)

Definition 2.1 (Differential Ring) A ring R is said to be a *differential ring* if there exists a differential operator from R to R, i.e., a map $d: R \to R$ such that, for all α and β in R:

- d is *linear*, i.e. $d(\alpha + \beta) = d(\alpha) + d(\beta)$;
- d satisfies the *product rule*, i.e. $d(\alpha\beta) = d(\alpha)\beta + \alpha d(\beta)$. □

For instance, the ring of analytic functions over a domain of $\mathbb{C}$ is a differential ring.

Definition 2.2 (Differential Ideal) A subset I of a differential ring R is a *differential ideal* if it is an algebraic ideal of R and moreover, it is closed under the d operator, i.e. if $d(I) \subseteq I$. □

If S is a subset of R and I is the minimal differential ideal of R containing S, then S is said to be a system of generators for I, or equivalently I is said to be the ideal generated by S. If $S = \{f_1, \ldots, f_n\}$ then I is denoted by $[f_1, \ldots, f_n]$.

[1] With unity.

Since R is a particular algebraic ring one can also consider the algebraic ideal J generated by S. This ideal is sometimes denoted by $(f_1, \ldots, f_n)$. Note that, in general, $(f_1, \ldots, f_n) \neq [f_1, \ldots, f_n]$.

Differential rings of particular interest are the ones constructed from a differential ring R by adjoining some differential indeterminates, as follows.

Definition 2.3 (Ring of Differential Polynomials) Let R be a differential ring. Consider the ring of polynomials $A = R[x_0, x_1, \ldots, x_n, \ldots]$ with a denumerable number of variables. A is a differential ring once the derivation d' on R is extended to a derivation d on A:

- $d(r) = d'(r)$ for all the elements $r \in R$;
- $d(x_i) = x_{i+1}$ for $i \geq 0$.

After renaming x_j as $d^{(j)}x$ (the derivation of order 0 is assumed to be the identity map), A can be denoted by $R[x, dx, d^{(2)}x, \ldots, d^{(n)}x, \ldots]$ or by $R\{x\}$; it is called the *ring of differential polynomials* in x over R.

It is possible to iterate the definition above to adjoin more variables to R, obtaining the differential ring $R\{x_1, \ldots, x_n\}$ of the differential polynomials in $x_1, \ldots, x_n$ over R. $\square$

2.1 Membership Problem

The most *fundamental problem of differential algebra* is the following:

Problem 2.1 (Ideal Membership Problem) *Let $I \subseteq R\{x_1, \ldots, x_n\}$ be a differential ideal in a ring of differential polynomials over a differential ring R and $f \in R\{x_1, \ldots, x_n\}$, a differential polynomial.*

Is there an effective procedure to decide if f is a member of the ideal I, i.e. $f \in I$? $\square$

The problem remains open; neither the solvability nor unsolvability of the problem has been demonstrated in spite of many years of concentrated effort in several promising directions. Clearly, because of its fundamental nature, we would like to obtain a better understanding of this problem. We discuss some of the approaches based on the H-bases ideas.

We start by discussing some differences between commutative and differential algebras, that are manifested by such facts as failure of a Hilbert-basis like theorem (only a weaker version, *Ritt-Raudenbusch Basis Theorem*, holds), existence of non-recursive differential ideals, etc. These differences warn us against the trappings of an effort based on a straightforward generalization of the existing ideas in the domain of commutative algebra.

Consider the ring of the integers $\mathbb{Z}$. It can be thought of as a differential ring with the trivial derivation d, i.e., the derivation satisfying: $d(m) = 0$ for any $m \in \mathbb{Z}$. Let $\mathbb{Z}\{x\}$ be the ring of differential polynomials in one indeterminate x over $\mathbb{Z}$.

Assume that the symbol $d^{(0)}$ denotes the identity map over $\mathbb{Z}\{x\}$. Define the differential polynomials f_i as follows:

$$f_i = \left(d^{(i)}x\right)^2 \qquad\qquad i \geq 0.$$

The following theorem has been shown in [17].

Theorem 2.1 *Let S be a subset of $\mathbb{N} \cup \{0\}$ and I_S, the differential ideal generated by the set $\{f_i : i \in S\}$. Then*

$$f_j \in I_S \quad \Leftrightarrow \quad j \in S.$$

In particular, if S is a nonrecursive subset of $\mathbb{N} \cup \{0\}$ then there is no algorithm to decide if a given differential polynomial f is in the ideal I_S. $\Box$

Thus we need to modify our "differential ideal membership problem" by restricting the class of ideals under consideration to only the recursive ones. A special case of such ideals would be finitely generated differential ideals, as all our ideals may be expected to be presented to a computer by some finitary means.

One reasonable attempt would be to use such classical tools as H-bases. We begin with some definitions:

Definition 2.4 (Weight Function) Let M denote the set of all monomials in the differential ring $R\{x_1, \ldots, x_n\}$. Consider the map $w\colon M \to \mathbb{R}$ so defined that:

- $w(x_i) = m_i$ with $m_i > 0$ for $i = 1, \ldots, n$;

- $w(d^{(k)}x_i) = (k + m_i)$ for any integer $k > 0$ and for $i = 1, \ldots, n$;
- For any monomial $m \in M$, $w(m) = \sum_i w(f_i)$ where f_i's range over the factors of m containing a single indeterminate or a derivative.

The function w is called a *weight* of the differential power product; the weight of a differential polynomial is the maximum weight of its power products.

A differential polynomial whose monomials have all the same weight, is called *isobaric*. The isobaric component of maximum weight is called *leading isobaric component* (or *head*) of f; it is denoted as $h(f)$. Notice that the isobaric components of a polynomial are not, generally, monomials, but isobaric polynomials. $\square$

Definition 2.5 A differential ideal I is called isobaric if, whenever a differential polynomial f is in I, all of its isobaric components belong to I. $\square$

In a fashion similar to the case of homogeneous ideals of polynomial rings, it can be shown that a differential ideal I in a ring of differential polynomials with *constant coefficients* is isobaric if and only if it has a system of isobaric generators. However, the above statement, in general, is not true if the derivatives of the coefficients of the polynomials are not zero. In fact in this case the derivative of an isobaric polynomial may not be isobaric. (See [17] for some examples and discussions.)

2.2 H-bases of Differential Ideals

Now, we propose the following generalization of the H-bases to differential ideals in rings of differential polynomials with coefficients in a computable differential field of characteristic zero. While the structure of the proposed H-bases is a natural extension of the classical one, it is not clear whether it is effective: for instance, we cannot simply generalize the usual H-bases algorithm and then rely on the standard "ascending chain condition" to prove termination. Thus we may ask the following questions:

- Do finitely generated differential ideals have finite sized H-bases?
- Do recursively generated differential ideals have recursive H-bases?

The answer to the first question is in the negative and not much is known about the second question and is a subject of current investigation.

Consider the ring $A = K\{x_1, \ldots, x_n\}$ together with a weight function w defined on it. A is isomorphic to the direct sum $\oplus_{i=0}^{\infty} A_i$, where A_i is, for any integer i, the finite dimensional K-vector space of the isobaric polynomials of weight i.

Definition 2.6 Let I be a differential ideal in $K\{x_1, \ldots, x_n\}$. Consider the *algebraic* ideal $H(I)$, generated by the set of all the leading isobaric components of the differential polynomials in I.

Similarly, given a basis S for the differential ideal I, consider the *algebraic* ideal $H(S)$ generated by the leading isobaric components of the polynomials in the set $\overline{S} = \{g : g = d^{(k)}s \text{ with } s \in S, \quad k \in \mathbb{N}\}$. Generally, $H(S)$ is properly contained in $H(I)$.

The basis S of the differential ideal I is said to be an *H-basis* if

$$H(S) = H(I).$$

The property of being an H-basis depends on the weight function considered over the ring. $\square$

Note that if K is *not a field of constants*, derivation and extraction of the leading isobaric components do not, in general, commute. Then the algebraic ideals $H(I)$ and $H(S)$ may not be differential ideals.

To get a 'differential' description of H-bases, in this case, one may adopt the point of view of [38] and [39]. Namely, one introduces a new derivation d_* on the ring A such that $d_*(K) = \{0\}$. Otherwise d_* is equal to d. See [17] and [38] for an extensive discussions of these questions.

Let S be a set of differential polynomials such that for any fixed integer k there are only finitely many elements in S with weight k. Now consider the following variant of "*rewriting procedure*" for differential polynomials:

Let f be an isobaric differential polynomial. Suppose that $f_1, \ldots, f_p$ are the elements of S, each of weight $w(f)$: Their leading isobaric components generate a subspace V of the vector space W of all the isobaric polynomials of weight $w(f)$. Because the field of the coefficients is computable it is possible to find $h_1(f)$ and $h_2(f)$ such that $h(f) = h_1(f) + h_2(f)$, with $h_1(f)$ in V and $h_2(f)$ in the orthogonal complement of V. In particular, it is possible to compute elements a_i in K such that

$$h_1(f) = a_1 h(f_1) + \cdots + a_p h(f_p).$$

The differential polynomial

$$\hat{f} = f - a_1 f_1 + \cdots + a_p f_p$$

is said to be a *reduct* of f modulo S. The "*reduction*" relation is denoted by the following notation:

$$f \to^S \hat{f}.$$

The reduction process can be generalized to an arbitrary differential polynomial g, by simply applying it to each one of the isobaric components of g.

A simple argument shows that no polynomial f can lead to an infinite chain of reductions.

Proposition 2.2 *Let I be a differential ideal; S, an H-basis for I, and f, a differential polynomial. Then f is in I if and only if any maximal chain of reductions with respect to S ends with* 0. □

It has also been shown in [17] that

Theorem 2.3 *Let I be a differential ideal of the ring $K\{x_1, \ldots, x_n\}$, with a recursive set of generators S of isobaric differential polynomials. Then*

- *S is an H-basis of the ideal I.*
- *I is a recursive subset of the ring $K\{x_1, \ldots, x_n\}$. That is I has an effective membership algorithm.* □

For other similar results also see [5] and [38]. However, we still do not have a complete algorithm. There are several problems: not all differential ideals are *isobarizable*—that is one cannot always choose a weight function that will make a given differential ideal isobaric; not all recursively generated isobaric differential ideals may have a finite computable H-bases. We present a counter-example due to Olivier that exhibits a finitely generated differential ideal without a finite H-basis.

Let $\mathbb{F}$ be a differential field of characteristic zero and $\mathbb{F}\{x,\, y\}$, a ring of differential polynomials in the variables x and y. Let M

$$m = [dx - 1, x\,dy + y]$$

be a finitely generated prime ideal in $\mathbb{F}\{x,\, y\}$. Now considering the algebraic ideal $H(M)$, we see that it is generated by $\{dx,\, x\,dy\} \cup J$, where J is an ideal in $\mathbb{F}\{y\}$ without a finite set of generators. Thus M, although finitely generated, has no finite H-basis. For more details, consult [15]. Thus, at present, the differential ideal membership problem remains an intriguing open problem of utmost fundamental interest.

2.3 Characteristic Sets

Let $K\{x_1,\, \ldots,\, x_n\}$ denote, as before, the ring of differential polynomials in n variables, with coefficients in an algebraic field K of characteristic zero. Consider a fixed ordering on the set of variables; without loss of generality, we may assume that the given ordering is the following:

$$x_1 < x_2 < \cdots < x_n.$$

Definition 2.7 (Class) Let $f \in K\{x_1, \ldots, x_n\}$ be a multivariate differential polynomial with coefficients in K. A variable x_j is said to be *effectively present in* f if some (differential) monomial in f with nonzero coefficient contains a (strictly) positive power of $d^{(i)}x_j$ $(i \geq 0)$.

The *class* of a differential polynomial $f \in K\{x_1,\, \ldots,\, x_n\}$ with respect to a given ordering is defined as follows:

1. If *no variable* x_j *is effectively present in* f, (i.e., $f \in K$), then, by convention, $\mathrm{Class}(f) = 0$.

2. Otherwise, if x_j *is effectively present in* f, *and no* $x_i > x_j$ *is effectively present in* f (i.e., $f \in k[x_1,\, \ldots,\, x_j] \setminus k[x_1,\, \ldots,\, x_{j-1}]$), then $\mathrm{Class}(f) = j$. $\square$

Given a differential polynomial f and a variable x_j effectively present in f, $\mathrm{ord}_{x_j}(f)$ (*order of* f *with respect to* x_j) denotes the greatest i such that $d^{(i)}x_j$ is effectively present in f.

Similarly, given a differential polynomial f and the k^{th} derivative of a variable x_j [i.e., $d^{(k)}x_j$] effectively present in f, $\deg_{d^{(k)}x_j}(f)$ (*degree of f with respect to* $d^{(k)}x_j$) denotes the greatest i such that $(d^{(k)}x_j)^i$ is effectively present in f.

Definition 2.8 (Ordering on the Differential Polynomials)
Given two differential polynomials f_1 and $f_2 \in K\{x_1, \ldots, x_n\}$, we say f_1 is of *lower rank than* f_2,

$$f_1 \prec f_2,$$

if

1. $\text{Class}(f_1) < \text{Class}(f_2)$, or
2. $\text{Class}(f_1) = \text{Class}(f_2) = j$ and $\text{ord}_{x_j}(f_1) < \text{ord}_{x_j}(f_2)$, or
3. $\text{Class}(f_1) = \text{Class}(f_2) = j$, $\text{ord}_{x_j}(f_1) = \text{ord}_{x_j}(f_2) = k$ and $\deg_{d^{(k)}x_j}(f_1) < \deg_{d^{(k)}x_j}(f_2)$.

Note that there are distinct differential polynomials f_1 and f_2 that are not comparable under the preceding order. In this case, f_1 and f_2 are said to be of the *same rank*, $f_1 \sim f_2$. $\square$

If f_1 is of class $j > 0$, then f_2 is said to be *reduced with respect to* f_1 if f_2 is of lower rank than f_1 in x_j, i.e., if

1. $\text{ord}_{x_j}(f_1) > \text{ord}_{x_j}(f_2)$, or
2. $\text{ord}_{x_j}(f_1) = \text{ord}_{x_j}(f_2) = k$ and $\deg_{d^{(k)}x_j}(f_1) > \deg_{d^{(k)}x_j}(f_2)$.

Definition 2.9 (Ascending Set) A sequence of differential polynomials $\mathcal{F} = \langle f_1, f_2, \ldots, f_r \rangle \subseteq K\{x_1, \ldots, x_n\}$ is said to be an *ascending set* (or *chain*), if one of the following two conditions holds:

1. $r = 1$ and f_1 is not identically zero;
2. $r > 1$, and $0 < \text{Class}(f_1) < \text{Class}(f_2) < \cdots < \text{Class}(f_r) \le n$, and each f_i is reduced with respect to the preceding differential polynomials, f_j's $(1 \le j < i)$.

Clearly, every ascending set is finite and has at most n elements. □

Definition 2.10 (Ordering on the Ascending Sets) Given two ascending sets

$$\mathcal{F} = \langle f_1, \ldots, f_r \rangle \quad \text{and} \quad \mathcal{G} = \langle g_1, \ldots, g_s \rangle ,$$

we say $\mathcal{F}$ is of *lower rank* than $\mathcal{G}$, $\mathcal{F} \prec \mathcal{G}$, if one of the following two conditions is satisfied,

1. There exists an index $i \leq \min\{r, s\}$ such that
$$\Big(\forall\, 1 \leq j < i\Big) \Big[f_j \sim g_j\Big] \text{ and } \Big[f_i \prec g_i\Big];$$
2. $r > s$ and $\Big(\forall\, 1 \leq j \leq s\Big) \Big[f_j \sim g_j\Big]$.

Note that there are distinct ascending sets $\mathcal{F}$ and $\mathcal{G}$ that are not comparable under the preceding order. In this case $r = s$, and $(\forall\, 1 \leq j \leq s)\ [f_j \sim g_j]$, and $\mathcal{F}$ and $\mathcal{G}$ are said to be of the *same rank*, $\mathcal{F} \sim \mathcal{G}$. □

Note that the family of ascending sets endowed with the ordering "$\prec$" is a well-ordered set.

Definition 2.11 (Characteristic Set) Let I be a differential ideal in $K\{x_1, \ldots, x_n\}$. Consider the family of all ascending sets, each of whose components is in I,

$$\mathbf{S}_I = \Big\{\mathcal{F} = \langle f_1, \ldots, f_r \rangle \ :\ \mathcal{F} \text{ is an ascending set} \\ \text{and } f_i \in I, 1 \leq i \leq r\Big\}.$$

A minimal element in $\mathbf{S}_I$ (with respect to the $\prec$ order on ascending sets) is said to be a *characteristic set* of the differential ideal I. □

Let $f \in K\{x_1, \ldots, x_n\}$ be of class j and of order k in x_j. Let u denote $d^{(k)}x_j$. The *separant* of f is the differential polynomial $\frac{\partial}{\partial u} f$. The coefficient of the highest power of $d^{(k)}x_j$ in f will be called *initial* of f. The separant and initial of f will be denoted by $S(f)$ and $I(f)$ respectively.

For a given set $\mathcal{F} = \{f_1, \ldots, f_r\} \subseteq K\{x_1, \ldots, x_n\}$ of r differential polynomials, let us write $H_{\mathcal{F}}$ to denote

$$H_{\mathcal{F}} = \prod_{i=1}^{r} I(f_i)S(f_i),$$

where $I(f_i)$ and $S(f_i)$ are the initial and separant of f_i. Now, for the differential ideal $[\mathcal{F}]$, note that the following is also a differential ideal:

$$[\mathcal{F}] : H^\infty = \left\{g : \Big(\exists j \geq 0\Big) \ \Big[H_{\mathcal{F}}^j \in [\mathcal{F}]\Big]\right\}.$$

Note that if I is a differential ideal and $\mathcal{F}$ is a characteristic set of I then, it can be shown ([45]) that

$$[\mathcal{F}] \subseteq I \subseteq [\mathcal{F}] : H^\infty.$$

Furthermore if I is prime then

$$I = [\mathcal{F}] : H^\infty.$$

Because of these properties, characteristic sets do ameliorate the problem of not being explicitly construct "nice" bases for differential ideals, in general.

The characteristic sets are ideal tools for and were originally developed to address the following problems:

Problem 2.2 (Solvability Problem) *Given a set of differential polynomials* $\{f_1, \ldots, f_r\} \subseteq K\{x_1, \ldots, x_n\}$.

Is the system of differential polynomial equations

$$f_1 = 0, f_2 = 0, \ldots, f_r = 0,$$

consistent? I.e., Do the equations have any solution? □

A related problem is that of determining if every solution of a system of differential polynomial equations

$$f_1 = 0, f_2 = 0, \ldots, f_r = 0,$$

is also a solution of another differential equation $g = 0$.

Characteristic set based techniques have also found applications in theorem proving [7, 55, 56, 57], elimination theory with applications in control theory [14, 11] and other areas where there is no standard bases related techniques.

However, there are several open problems related to this area and worth investigating. Not much is known about the efficiency and complexity analysis of these algorithms. In fact, even for the algebraic case, until very recently there was no effective algorithm to compute a characteristic set or to analyze its algebraic and algorithmic complexity [15, 16]. Also, it is worth investigating whether these techniques can be used in providing effective representation of differential ideals. Following open problem is of fundamental interest (also, see [49]):

Problem 2.3 (Prime Decomposition Problem) *Given a radical differential ideal $J \subseteq K\{x_1, \ldots, x_n\}$.*

Find the minimal prime components $\{P_1, \ldots, P_p\}$ of J.

$$J = P_1 \cap P_2 \cap \cdots \cap P_p. \quad \square$$

2.4 Formal Solutions of Differential Equations

In addition to studying the ideal-theoretic and solvability issues for a system of differential (polynomial) equations, one is also commonly interested in understanding the solutions, their structures or the symmetries involved in the equations. One area of rather vigorous investigation has been in obtaining "formal expressions that represent solutions of differential equations." ([49]). Such solutions are either explicitly given by formal power series, Liouvillian functions, error functions etc., or implicitly given by elementary first integrals and Lie-theoretic techniques.

The easiest problem in this area is that of solving the differential (order 1 and linear) equation:

$$\frac{d}{dx}g - f(x) = 0.$$

We ask when a solution $g(x) = \int f(x)\, dx$ can be expressed in terms of elementary functions (functions such as log, exp, sin, arcsin, etc. of elementary calculus). This is the integration problem and was briefly considered in the introduction. There is a considerable literature for this problem and is relatively well-understood ([4, 9, 43, 49]).

Next one considers the problem of higher order but still linear differential equations:

$$\mathcal{L}(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y + a_0 = 0. \quad (1)$$

While the question is rather simple for the case when a_i's are constants, it turns into a rather difficult problem when one assumes that $a_i \in \mathbb{F}$ ($\mathbb{F} =$ a differential field containing an algebraically closed subfield of constants of characteristic zero). More concretely, one asks when the equation $\mathcal{L}(y) = 0$ has Liouvillian solutions, and if it does have such solutions, whether there are effective algorithms to find the solutions. Kovacic [30] dealt with the case when the order is two. The general case has been addressed by Singer[48].

In order to understand the techniques here, one has to rely on the "*Differential Galois Theory*," a non-trivial generalization of the corresponding algebraic counterpart. Consider two differential field $(\mathbb{F}_1, d_1)$ and $(\mathbb{F}_2, d_2)$, equipped with their corresponding derivation operations d_1 and d_2, respectively. Here, one assumes that the constants of such a differential field is a subfield of characteristic 0 and algebraically closed. Two such fields are *differentially isomorphic* if there is a field isomorphism $\Phi : \mathbb{F}_1 \to \mathbb{F}_2$ such that $\Phi \circ d_1 = d_2 \circ \Phi$. A differential isomorphism of a field to itself is a *differential automorphism*.

A differential field $(\mathbb{F}', d')$ is a *differential (field) extension* of $(\mathbb{F}, d)$ if $\mathbb{F} \subset \mathbb{F}'$ and d' restricted to $\mathbb{F}$ coincides with d. Let $u_1, \ldots, u_r \in \mathbb{F}'$ and $\mathbb{F} \subset \mathbb{F}'$ is a differential subfield of $\mathbb{F}'$ (both equipped with the same derivation d). Then the smallest differential field containing $\mathbb{F}$ and the elements $u_1, \ldots, u_r$ is denoted $\mathbb{F}\{u_1, \ldots, u_r\}$.

Assume that $y_1, y_2, \ldots, y_n$ are solutions of the order n linear differential equation of Eq. (1) that are *linearly independent* over the subfield of constants $K_{\mathbb{F}}$ of $\mathbb{F}$. (I.e., the Wronskian determinant, $W(y_1, \ldots, y_n)$, does not vanish.) Consider the field extension $\mathbb{F}' = \mathbb{F}\{y_1, \ldots, y_n\}$—such an extension is called a *Picard-Vessiot extension* of $\mathcal{L}(y) = 0$ (Eq. (1)) if no new constant is introduced into the differential extension. Kolchin has shown that under the assumptions of this section, to every linear differential equation there exists a Picard-Vessiot extension, which is unique up to differential isomorphism.

Also, given a Picard-Vessiot extension of $\mathcal{L}(y) = 0$, $\mathbb{F}' = \mathbb{F}\{y_1, \ldots, y_n\}$, the *differential Galois group* of $\mathcal{L}(y) = 0$ ($\mathrm{Gal}_d(\mathbb{F}'/\mathbb{F})$) is the group of all differential automorphisms $\Phi : \mathbb{F}' \to \mathbb{F}'$ such that $\Phi f = f$ for all $f \in \mathbb{F}$. Then it has been shown (see Singer[48]) that $\mathcal{L}(y) = 0$ is solvable in terms of Liouvillian functions (i.e. its Picard-Vessiot extension lies in a Liouvillian extension of $\mathbb{F}$) if and only if its Galois group ($\mathrm{Gal}_d(\mathbb{F}'/\mathbb{F})$) contains a solvable (in the algebraic sense) subgroup of finite index.

However, finding the Liouvillian solution is still hard and one attempt is to find these solutions by effectively searching over a bounded space (see Singer[47]). From the solvable subgroup H of $\mathrm{Gal}_d(\mathbb{F}'/\mathbb{F})$ and the poles of the coefficients of $\mathcal{L}(y) = 0$, one can obtain bounds N and P such that if $\mathcal{L}(y) = 0$ has only Liouvillian solutions then it has a solution y such that $u = y'/y$ satisfies an algebraic equation of degree bounded by N and P. There are several effective algebraic algorithms to compute such a u, and then we have the solution $y = ze^{\int u}$ where z is found by solving the lower order equation $\mathcal{L}^*(z) = 0$, obtained by the change of variable: $y = ze^{\int u}$. By proceeding this way, we can effectively compute all the Liouvillian solutions. For more details, see Singer[47].

While this is effective, an explicit complexity analysis of this method of solution remains to be performed. Also, one needs to explore various other techniques to improve the complexity of this problem.

However, the situation deteriorates as one begins to consider non-linear differential equations. Except for certain special cases, one still does not have an adequate answer for the following problem (see Singer[49]):

Problem 2.4 (Formal Solutions of Nonlinear Differential Equations)
Given a polynomial first order differential equation:

$$f(x, y, y') = 0.$$

Decide if $f = 0$ has an elementary solution. If it does have one, find it. □

Clearly, there are many other interesting problems of this nature, involving systems of linear and nonlinear differential equations, partial differential equations, etc. Some negative results and in some special cases, few positive results are known here. But, it is our impression that much more needs to be done, before this field is ready to be used by practitioners in a mundane manners.

3 Applications

Clearly, since the formulation of physical laws by Newton using the concepts of calculus, mathematical physics, applied mathematics and more recently, mathematical approaches in social sciences (e.g. economics) have all been based on models that are described by ordinary or partial differential equations and the evolution of the object modeled is described by the solution to these equations. More recently, beginning in the last century, the field of control theory has begun to look at the synthesis of physical systems (via a feedback control of an autonomous plant) that exhibit certain desired behavior. While most of the success in this field are in the case when the underlying plant is finite-dimensional and linear, there is currently much interest in extending the theory to non-linear situations. One reason for this interest is the emergence of the field of robotics, where the dynamics of the system as well as the kinematic and dynamical constraints require one to study symbolic solutions of much more general systems of differential equations than what is common in classical control theory.

We begin by discussing the problem of *nonholonomic motion planning* which arises in robotics and then proceed to touch on some related problems in control theory as well as the simulation theory.

3.1 Robotics

In robotics the problem of *motion planning* is to navigate a robot (in general, a collection of robots) from an initial configuration to a final configuration while obeying a set of constraints (kinematic as well as dynamic) imposed externally by a set of obstacles, a set of contact conditions or by virtue of some conservation laws.

The best way to study this problem is in the so-called *configuration space*, $\mathcal{C}$ (or *C-Space*), which is a parameter space corresponding to all the potential *configurations of the robot*. Clearly, the set of physical obstacles in the robot's environment render certain subsets of the C-space infeasible—this subset is called the *forbidden space* and its complement the *free space*. About a decade ago it was observed by several researchers that if the obstacles are algebraic (i.e., can be described by piecewise polynomial surfaces) then the free space (as well as forbidden space) are semialgebraic and in the absence of any other constraints (specifically, differential), the motion planning problem reduces to determining if two given points are in the same connected component of the free space. In such a situation, one can use some ideas from computational semialgebraic geometry to devise algorithmic solutions. Since in a semialgebraic set path connectivity implies semialgebraic path connectivity, one can in fact generate a semialgebraic (thus, continuous) path connecting the initial configuration to the final one. This classic situation is usually studied under the title of *holonomic motion planning*.

However, the situation gets more complicated if there are additional constraints on the robot path which are described by certain local conditions that can be described as *path constraints* (as opposed to point-wise constraints) and thus by differential equations or inequations. For instance, at certain configuration the constraints may dictate that motions in only certain directions are allowed. Another way of describing these constraints may involve high dimensional *phase spaces* (e.g., spaces consisting of robot configuration parameters as well as velocities, accelerations etc.)

Usual examples in robotics where this sort of *nonholonomic motion planning* problems play a central role include: *articulated robots*—the joints impose constraints on the relative motions of the associated links; *mobile robots*—where the robot is allowed to move in certain directions (e.g., front-wheel drive cars that can drive and steer but not move side-wise); *space robotics*; *motion in contact*—fingers manipulating an object without breaking the contacts or sliding. See [3, 12, 19, 21, 31, 32, 36, 37].

In this context, one addresses the following three questions:

- *When are the local constraints local?* In other words, given a set of nonholonomic constraints, how do we know that they are *actually nonholonomic*, (i.e., nonintegrable)? This is the *Characterization Problem.*

- *How do the set of constraints (both holonomic and nonholonomic) restrict the space of configurations reachable from an initial configuration?* This is the *Controllability Problem.*

- *Given a robot subject to holonomic and nonholonomic constraints, how can we plan a path for the robot to go from an initial configuration to a final configuration?* This is the *Path Planning Problem.*

More formally, we are given the following:

- **Robot:** A—assumed to be in motion.
- **Phase Space:** $\mathcal{P} = \mathcal{C} \times \mathcal{T}_Q(\mathcal{C}) =$

$$= \Big\{(q_1, \ldots, q_m, \dot{q}_1, \ldots, \dot{q}_m) : q_i\text{'s are the configuration coordinates and } \dot{q}_i\text{'s the velocity coordinates}\Big\}.$$

  Here, $Q \in \mathcal{C}$ is a configuration in the configuration space and $\dot{Q} \in \mathcal{T}_Q(C)$ is a velocity in the tangent space of $\mathcal{C}$ at Q.

- **Constraints:** We assume that the robot A satisfies the following scalar constraint:

$$G(Q, \dot{Q}, t) = G(q_1, \ldots, q_m, \dot{q}_1, \ldots, \dot{q}_m, t) = 0. \tag{2}$$

If the kinematic constraint given by the equation (2) is non-integrable (i.e. $\dot{q}_1$, ..., $\dot{q}_m$ cannot be eliminated from the equation (2)) then the constraint is a *nonholonomic constraint*.

Example 3.1 Following is a classic example involving the *front-wheel drive car*. (See Figure 3.1.)

Rolling constraints on the wheel give rise to the following non-holonomic constraints:

$$\frac{\dot{x}}{\cos\theta} = \frac{\dot{y}}{\sin\theta} = \frac{\dot{\theta}}{\tan\phi}.$$

That is

$$\begin{aligned} \dot{x}\sin\theta - \dot{y}\cos\theta &= 0 \\ \dot{x}\tan\phi - \dot{\theta}\cos\theta &= 0. \end{aligned}$$

If one introduces the following *control variables*

$$\begin{aligned} u_1 &= \textit{driving velocity} \\ u_2 &= \textit{steering velocity}, \end{aligned}$$

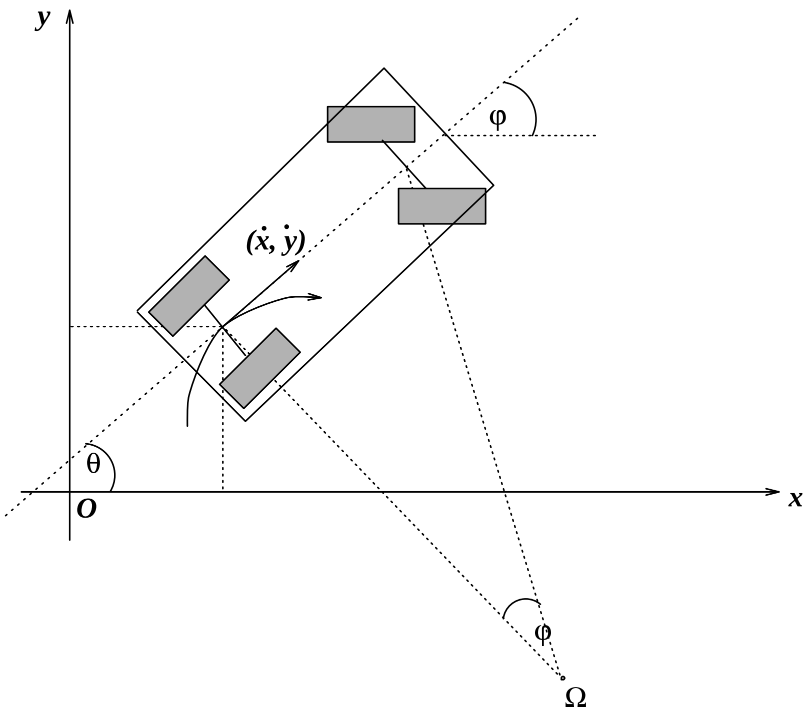

Figure 1: **Front Wheel Drive Car**

then the above constraints can also be rewritten as follows:

$$\begin{aligned} \dot{x} &= \cos\theta\, u_1 \\ \dot{y} &= \sin\theta\, u_1 \\ \dot{\theta} &= \tan\phi\, u_1 \\ \dot{\phi} &= u_2. \end{aligned}$$

In the matrix notation, this is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \tan\phi \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u_2. \quad \Box$$

Now assume that the constraints are given by the following (differential) equality constraint:

$$W_1(Q)\dot{q}_1 + W_2(Q)\dot{q}_2 + \cdots + W_m(Q)\dot{q}_m = 0,$$

where $W_i : \mathcal{C} \to \mathbb{R}^k$ is a map from the m-dimensional configuration space to a k-dimensional vector space. We write the associated matrix as

$$\mathcal{W}(Q) = (\ W_1(Q)\ W_2(Q)\ \cdots\ W_m(Q)\ )$$

and assume that it is of full rank $k < m$. Then there is a linear subspace $\Delta(Q) \subset \mathcal{T}_Q(\mathcal{C})$ such that $\dim \Delta(Q) = m - k$. Thus there is a vector $U = (u_1, u_2, \ldots, u_m) \in \Delta(Q)$ such that

$$\mathcal{W}(Q)U = W_1(Q)u_1 + W_2(Q)u_2 + \cdots + W_m(Q)u_m = 0.$$

That is, there exists an $(m-k)$-distribution Δ (given by $(m-k)$ independent vector fields) with a local basis $X_1(Q), \ldots, X_{m-k}(Q)$ spanning $\Delta(Q)$ for all Q.

$$\Big(\forall Q \in \mathcal{C}\Big)\ \Big[\text{Span}\ (X_1(Q), \ldots, X_{m-k}(Q)) = \Delta(Q)\Big].$$

Thus any admissible path must satisfy the following equation:

$$\dot{Q} = \sum_{i=1}^{m-k} X_i(Q)u_i, \quad \text{where} \ \ U \in \mathbb{R}^{m-k}.$$

Thus the W's define the forbidden motion directions and X's describe the free (i.e., feasible) motion directions.

The set of vector fields given by Δ is called a distribution. Now, consider a manifold $N \subseteq \mathcal{C}$ such that

$$\Big(\forall Q \in \mathcal{C}\Big)\ \Big[\Delta(Q) = \mathcal{T}_Q(N)\Big].$$

We call N an integral manifold of Δ. Now, we have a solution to the *characterization problem*. If the system is holonomic then the integral manifold N of the distribution Δ is given by a level surface:

$$h_1(Q) = 0, \ldots, h_{m-k}(Q) = 0, \quad \forall Q \in \mathcal{C}.$$

In order to understand the *controllability problem* we need to consider the following Lie derivative operation. Let (X, Y) be a pair of independent vector fileds in $\Delta(Q)$. At the configuration Q, consider the following cyclic motion depicted in figure 3.1: A motion in the direction of X followed by a motion

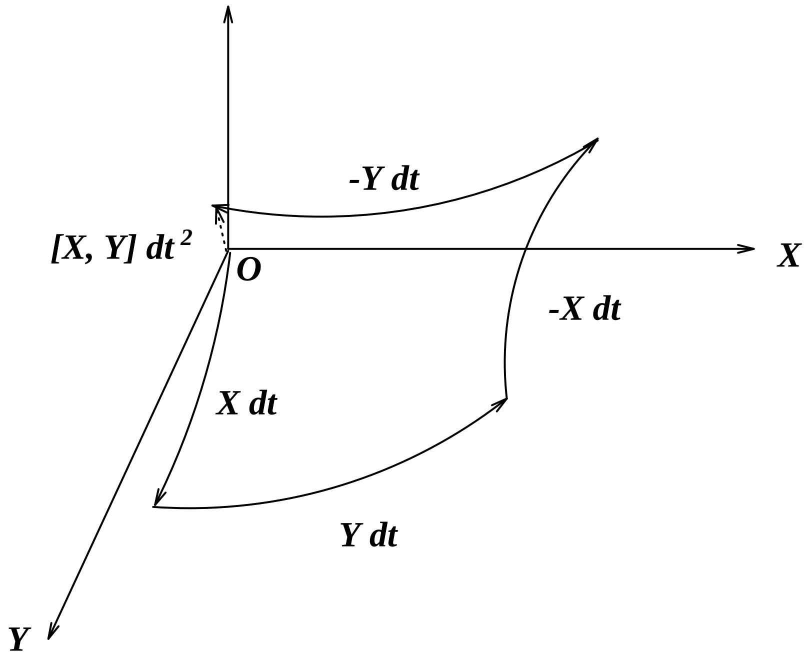

Figure 2: **Lie derivative**

in the direction of Y, then $-X$ and finally $-Y$. The resulting motion is then in the direction $[X, Y]$ (the Lie-bracket of X and Y) given by

$$\begin{aligned}
[X, Y] &= DY \cdot X - DX \cdot Y \\
&= \begin{pmatrix} \frac{\partial Y_1}{\partial q_1} & \cdots & \frac{\partial Y_1}{\partial q_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial Y_m}{\partial q_1} & \cdots & \frac{\partial Y_m}{\partial q_m} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \\
&\quad - \begin{pmatrix} \frac{\partial X_1}{\partial q_1} & \cdots & \frac{\partial X_1}{\partial q_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial X_m}{\partial q_1} & \cdots & \frac{\partial X_m}{\partial q_m} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}.
\end{aligned}$$

Example 3.2 Going back to our old example of the *front-wheel drive car*, we see that the distribution is given by the following two vectors: $X =$ drive and Y = steer.

$$X = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \tan\phi \\ 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

By Lie bracket operations, we can obtain additional directions: $[X,Y]$ = wriggle and $[X,[X,Y]]$ = slide.

$$[X,Y] \begin{pmatrix} 0 \\ 0 \\ -\sec^2\phi \\ 0 \end{pmatrix} \quad \text{and} \quad [X,[X,Y]] = \begin{pmatrix} -\sin\theta\sec^2\phi \\ \cos\theta\sec^2\phi \\ 0 \\ 0 \end{pmatrix}.$$

Note that $(X,\ Y,\ [X,Y],\ [X,[X,Y]])$ span the entire space and thus with drive, steer, wriggle and slide we can move a front-wheel drive car from any configuration to any other. $\square$

In general given a distribution

$$\Delta = \text{Span}\,(X_1, X_2, \ldots, X_m) = \Delta_0,$$

we can construct its *filtration* as follows:

$$\begin{aligned}
\Delta_0 &= \Delta \\
\Delta_1 &= \Delta_0 + [\Delta_0, \Delta_0] \\
&\qquad \text{where } [\Delta_0, \Delta_0] = \text{Span}\,([X,Y] : X \in \Delta_0, Y \in \Delta_0]) \\
&\vdots \\
\Delta_i &= \Delta_{i-1} + [\Delta_0, \Delta_{i-1}] \\
&\qquad \text{where } [\Delta_0, \Delta_{i-1}] = \text{Span}\,([X,Y] : X \in \Delta_0, Y \in \Delta_{i-1}]) \\
&\vdots
\end{aligned}$$

Assuming a regularity condition on the filtration (i.e., for all $Q \in \text{Nbhd}(Q_0)$, we have $\text{rank}(\Delta_i(Q)) = \text{rank}(\Delta_i(Q_0)))$ we see that

$$\Delta_0 \subseteq \Delta_1 \subseteq \cdots \subseteq \Delta_i \subseteq \cdots$$

and that $\text{rank}(\Delta_{i+1}) \geq \text{rank}(\Delta_i)$ and if $\text{rank}(\Delta_{p+1}) = \text{rank}(\Delta_p)$ then

$$\text{rank}(\Delta_p) = \text{rank}(\Delta_{p+1}) = \text{rank}(\Delta_{p+2}) = \cdots.$$

The smallest such p is called *degree of nonholonomy* of the distribution Δ. If $p = 0$ then the system is *involutive* and by Forbenius integrability theorem it is actually holonomic. Thus, once we compute a filtration, we have solved the *characterization problem*. Furthermore, if $p > 0$ then the system is nonholonomic, and if additionally $\text{rank}(\Delta_p) = m$ is the dimension of the configuration space $\mathcal{C}$ then the system is *maximally nonholonomic* and by Chow's theorem[8] the system is locally controllable (i.e., for any two configurations Q_i and Q_f in an open set of the configuration space $\mathcal{C}$ there is a path connecting Q_i and Q_f which obeys the nonholonomic constraints imposed by the distribution.) Thus, we also have a solution to the *controllability problem*.

This brings us to the final problem of *Path Planning*. Much less is known for this problem and it is currently an area of active research. Below, we summarize some of the main results involving the path planning problem.

1. Sussman and his colleagues have shown that if the system is nilpotent or nilpotentizable then assuming a regular filtration, the path planning problem can be solved using only piecewise constant control involving only finitely many discontinuities the number of which is a function of the degree of nilpotency. See [31, 53].

2. If the underlying system is triangular, then Sastry and his students have presented several path planning solutions for these special cases. [36, 37].

3. There are several numerical approaches that generate approximate paths and are based on regularization techniques, highly oscillating control or averaging techniques. See[19, 53].

We plan to understand and unify the underlying algebraic structures and devise solutions for more general cases involving drift, inequalities and algebraic constraints imposed by the obstacles or limiting constraints on the control.

A particularly interesting example of a system with drift arises in the following example, where one wishes to model and control a underwater vehicle. See for instance Tennant[54].

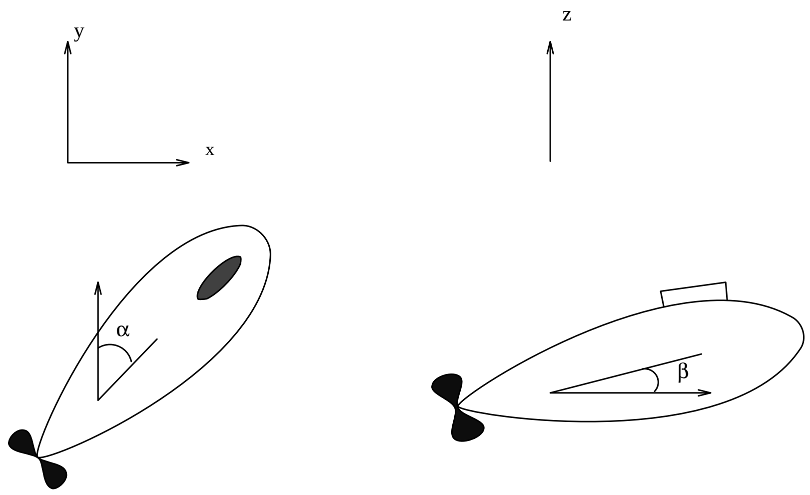

Figure 3: **A Underwater Vehicle**

Example 3.3 Consider the *underwater vehicle* as depicted in Figure 3.3.

The configuration of the vehicle is represented by the following column vector $X \in \mathbb{R}^3 \times S_2$:

$$\begin{pmatrix} x \\ y \\ z \\ \alpha \\ \beta \end{pmatrix},$$

where α is chosen to represent the direction in which the vehicle is pointed when projected into the horizontal xy-plane, and β is chosen to represent the angle that the vehicle makes with respect to horizontal. Traditionally, α is called the vehicle's *heading* and β its *pitch*. Together they parametrize the unit sphere S_2. The position of the vehicle (i.e., a reference point) is given by the position vector x, y and z.

Of course, the phase space of the entire vehicle is then given by a 10-dimensional space consisting of its configuration coordinates (as above) as well as 5 velocity coordinates $\dot{x}$, $\dot{y}$, $\dot{z}$, $\dot{\alpha}$ and $\dot{\beta}$. The linear velocity vector is then represented as

$$V = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}.$$

Since the motion is allowed only in the direction of V, the plane of disallowed velocity directions are then given by the plane spanned by the unit vectors

$$n_1 = \begin{pmatrix} \cos\alpha \\ -\sin\alpha \\ 0 \end{pmatrix} \quad \text{and} \quad n_2 = \begin{pmatrix} \sin\alpha\sin\beta \\ \cos\alpha\sin\beta \\ -\cos\beta \end{pmatrix}.$$

Now using the facts that

$$n_1 \cdot V = 0 \quad \text{and} \quad n_2 \cdot V = 0,$$

and proceeding as in the previous discussion, we find that the resulting constraints are given as

$$\begin{aligned} \dot{x} &= u' \sin\alpha\cos\beta \\ \dot{y} &= u' \cos\alpha\cos\beta \\ \dot{z} &= u' \sin\beta \\ \dot{\alpha} &= u_1 \\ \dot{\beta} &= u_2 \end{aligned}$$

where u_1 and u_2 are independent controls and $u' = v = |V|$ the magnitude of the linear velocity vector. Again by looking at the distribution associated with the system it is seen that the system is nonholonomic and further, by examining the filtration we see that it has a degree of holonomy is one.

In the above formulation, there is an unrealistic assumption that the magnitude of the velocity needs to be treated as a third independent control. However, by inertia considerations, we see that a more realistic model would in fact treat the velocity as part of the state and that the phase space should include an acceleration component $\dot{v}$. With these modifications, the resulting system looks as follows:

$$\begin{aligned} \dot{x} &= v \sin\alpha\cos\beta \\ \dot{y} &= v \cos\alpha\cos\beta \\ \dot{z} &= v \sin\beta \\ \dot{\alpha} &= u_1 \\ \dot{\beta} &= u_2 \\ \dot{v} &= u_3. \end{aligned}$$

Now the system has the following form:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\alpha} \\ \dot{\beta} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \sin\alpha\cos\beta \\ v\cos\alpha\cos\beta \\ v\sin\beta \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} u_2 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u_3.$$

Thus the first term represents a "drift" as zero control no longer implies zero motion.

Looking at the above formulation, it should be clear that given an initial configuration with a magnitude of initial velocity and a final configuration with a final velocity magnitude, the control problem is to move the submarine only by controlling the acceleration and the angular velocities for the heading and the pitch. The problem is rather hard, particularly, when additional constraints are placed on the magnitudes for the controls. These constraints roughly correspond to the limits on the allowable curvatures of the path and an upper bound on the allowable acceleration value. In his Master's thesis, Tennant ([54]) has provided an approximate solution to the control problem by projecting the problem into planar subcases and solving the planar problems exactly by an application of Dubin's theorem. However, the general problem still remains unsolved and is of utmost importance, especially when the underwater vehicles may need to deal with underwater obstacles. It should be noted that similar problems occur in many other instances, involving spacecrafts, airplanes and helicopters, for instance.

3.2 Control Theory

In control theory, a state space description of a plant is usually given by a system of differential equations. For instance, in the simplest possible (but widely used) formalism, a *linear continuous time time-invariant single-input-single-output (SISO)* system is described by a state equation and an output equation as follows:

$$\begin{aligned} \dot{x}(t) &= ax(t) + bu(t) \\ y(t) &= cx(t) + du(t), \end{aligned}$$

where the first equation describes the evolution of the state in the state-space and the last equation describes how the output depends on the current state and the control. Here, x is a state variable, u is a control variable and y is the output variable. One assumes that the input/control u and the output y are observable, but the state variable x is hidden or latent.

In general, one needs to consider more general description of either of the following two forms, the later being more general and preferable. The system in the *explicit form* is:

$$\begin{aligned} \dot{x}_1 &= f_1(X, u, \dot{u}, \ldots, u^{(m)}) \\ &\vdots \\ \dot{x}_n &= f_n(X, u, \dot{u}, \ldots, u^{(m)}) \\ y &= h(X, u) \end{aligned}$$

The system in the *implicit form* is:

$$\begin{aligned} f_1(\dot{x}_1, X, u, \dot{u}, \ldots, u^{(m)}) &= 0 \\ &\vdots \\ f_n(\dot{x}_n, X, u, \dot{u}, \ldots, u^{(m)}) &= 0 \\ h(y, X, u) &= 0. \end{aligned}$$

However, a large number of practical systems can be described easily by the *explicit form* and any specific algorithmic improvement one may be able to obtain for these cases are extremely valuable.

For instance, we saw that given a linear differential constraint equation of the following kind:

$$W_1(Q)\dot{q}_1 + W_2(Q)\dot{q}_2 + \cdots + W_m(Q)\dot{q}_m = 0,$$

by considering the local bases for the distribution $\Delta(Q)$, i.e.,

$$\Delta(Q) = \text{Span } (X_1(Q), \ldots, X_m(Q)),$$

we can write an associated control equation as follows:

$$\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_m \end{pmatrix} = X_1(Q)u_1 + X_2(Q)u_2 + \cdots + X_{m-k}(Q)u_{m-k}.$$

Since all the states in this case may be assumed to be observable (or measurable) by a set of sensors, we may further assume

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{pmatrix}.$$

Thus we may consider the nonholonomic motion planning problem simply a rather special case of the general control-theoretic problem. Thus, the ideas developed here (e.g., controllability, observability, choice of controls, etc.) are directly applicable to the situations described in the preceding section.

One of the major problems in control theory is to determine an *input-output relation* between the control inputs and the output variables starting from a redundant state-space description of the system. From an algebraic point of view, this is exactly the problem of *variable elimination* and comes under the subject of *elimination theory*. Thus all the theories related to *standard bases*, *characteristic sets* and *differential-algebraic resultants* play important roles.

This problem is also directly related to and the starting point for several other problems: determining whether two systems are isomorphic (i.e., if the two systems have the same input-output behavior), minimality of a state-space description, controllability and observability of a system. For additional discussions of these topics, consult [11, 14, 23, 24, 52].

The general approaches are as follows: Assume that the system (SISO) is described as given below. The general MIMO systems are handled in an identical manner.

$$\begin{aligned} \dot{x}_1 &= f_1(X, u, \dot{u}, \ldots, u^{(m)}) \\ &\vdots \\ \dot{x}_n &= f_n(X, u, \dot{u}, \ldots, u^{(m)}) \\ y &= h(X, u) \end{aligned}$$

Consider the following differential ideal I in the differential ring $K\{x,\, u,\, y\}$:

$$I = [\dot{x}_1 - f_1, \ldots, \dot{x}_n - f_n, y - h].$$

The input-output relation is then obtained by finding the contraction I^c of the ideal I to the ring $K\{u, y\}$. The generators of $I^c = I \cap K\{u, y\}$ give the differential polynomials involving only u and y.

Example 3.4 In the following example from Forsman[14], we consider two coupled tanks: one conical and the other cylindrical, where we wish to control the height of the fluid in the cylindrical container, z_2. See Figure 3.4.

The state-space equations for this system is:

$$\begin{aligned} \dot{z}_1 &= \frac{u - \sqrt{z_1}}{z_1^2} \\ \dot{z}_2 &= \sqrt{z_1} - \sqrt{z_2} \\ \\ y &= z_2, \end{aligned}$$

where u is the input flow and z_1 and z_2 are the height of the fluid in the conical and cylindrical tanks, respectively.

We can convert the system to a polynomial system by introducing two new variables x_1 and x_2, where

$$x_1^2 = z_1 \quad \text{and} \quad x_2^2 = z_2.$$

We next consider the differential ideal generated by the following systems of differential equations:

$$2x_1^5\dot{x}_1 + x_1 - u = 0 \quad 2x_2\dot{x}_2 + x_2 - x_1 = 0 \quad x_2^2 - y = 0.$$

After eliminating x_1 and x_2, we obtain the following input-output relation:

$$\begin{gathered} (20\dot{y}^8y^2 - 4\dot{y}^{10}y - 40\dot{y}^6y^3 + 40\dot{y}^4y^4 - 20\dot{y}^2y^5 + 4y^6)\ddot{y}^2 \\ + (4u\dot{y}^5y - 4\dot{y}^6y - 20\dot{y}^4y^2 + 40u\dot{y}^3y^2 + 20\dot{y}^2y^3 + 20u\dot{y}y^3 + 4y^4)\ddot{y} \\ - \dot{y}^2y^5 + 5\dot{y}^4y^4 - 10\dot{y}^6y^3 + 20u\dot{y}^3y^2 + 10\dot{y}^8y^2 + y^2 - 8\dot{y}^6y + 10u\dot{y}^5y \\ - u^2y + 2u\dot{y}y - \dot{y}^2y - 5\dot{y}^{10}y + \dot{y}^{12} + 8\dot{y}^2y^3 + 2u\dot{y}y^3 = 0. \quad \Box \end{gathered}$$

However, one problem with our formulation is that $I^c = I \cap K\{x, y\}$ may not have a finite basis even if I is given by a finite set of generators. This creates a problem that, unlike the purely algebraic case, cannot be solved using the standard bases approach. On the other hand, the approach suggested

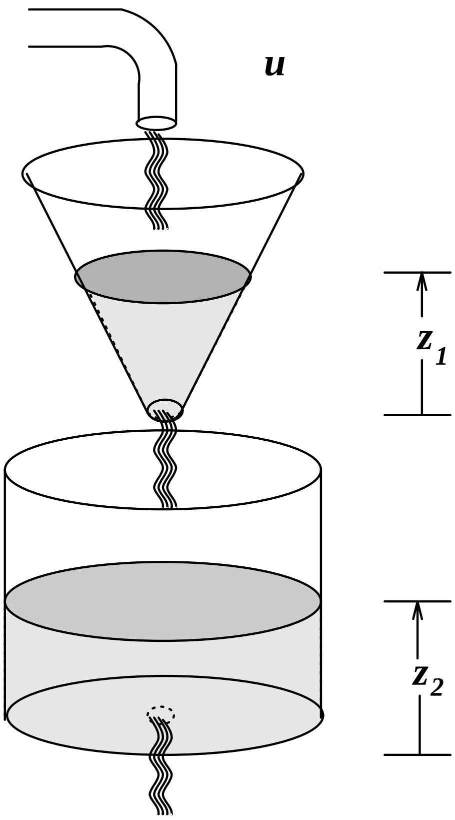

Figure 4: **The coupled tank example.**

by Ritt using the characteristic sets comes to our rescue. Here, the problem has to be reformulated in order that we agree to accept a relation between input-output as opposed to a description of the entire contracted ideal.

Another approach is via extended Lie-derivative operators. Let us define u_i and x_i for all $i \in \mathbb{N}$ to mean

$$u_i = \frac{d^i}{dt^i}u, \quad \text{and} \quad x_i = \frac{d^i}{dt^i}x.$$

Given f_1, ..., f_n as in the state-space equations, we define the *extended Lie-derivative operator* as

$$\mathcal{L}_f = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{\infty} u_{i+1} \frac{\partial}{\partial u_i}.$$

Then the input-output relation can be determined by considering the purely algebraic ideals of the form

$$(y - h, \dot{y} - \mathcal{L}_f h, \ldots, y^{(n)} - \mathcal{L}_f^n h) \cap K[u, \dot{u}, \ldots, u^{(m+n)}, y, \dot{y}, \ldots, y^{(n)}].$$

In this special case, one can use the classical approaches of purely algebraic elimination theory (e.g., Gröbner bases) to obtain the solution.

There is a need to systematically study the input-output relations and the related problems of isomorphism, controllability, observability and minimality of description, etc. Also, of interest is the study of the complexity of the problem both in the abstract sense as well as practical sense with specific application areas in mind.

3.3 Simulation

Another interesting application comes from the simulation of complex dynamical systems. One hopes to understand the structure of the system and its evolution by studying the paths of evolution of a dynamical system under dynamically changing environments. A large class of systems of interest are provided by the so-called "multi-body" systems encompassing such systems as astronomical systems (galactic and planetary systems), the human body, spacecrafts, molecular systems (proteins, inorganic molecules), vehicular systems and robots. The dynamics of these systems cannot be easily approximated by linear differential systems, since large motions are characteristic of all these systems. The inherent nonlinearity leads to the study of the differential algebraic systems that can be arbitrarily complex.

The main steps in the process of simulating these objects are essentially of two kinds: (1) derivation of a dynamic model of the system (for instance, Lagrangian or Newtonian mechanics) and (2) reduction of the model that takes into account the inherent symmetry in the system. The final model can then be presented in a suitably simplified form whose step-wise evolution can then be studied by numerical means.

For instance, if our interest is in a planar multi-body system then its configuration space is given by

$$\mathcal{C} = (S^1)^N \times \mathbb{R}^2,$$

where N is the number of rigid bodies involved in the system. The system can then be coordinatized on the tangent bundle $\mathcal{T}_Q$ by

$$(\theta_i, \ldots, \theta_N, \omega_1, \ldots, \omega_N),$$

the relative angles and the angular velocities. The Lagrangian in these coordinates is then given by

$$L = \frac{1}{2}\Omega' J(\Theta)\Omega + \frac{\|P\|}{2m},$$

where Ω is the vector of angular velocities, P is the linear momentum (of the center of mass of the system) and $J(\Theta)$ is the pseudo-inertia matrix of the system and depends on the relative angles. The symmetries of the system can then be recognized by studying the group actions of translational group ($\mathbb{R}^2$) and the rotational group (S^1) and the system dynamics can be appropriately reduced to get a final description for the simulation purposes. This approach is essentially what has been proposed by Sreenath and Krishnaprasad[50, 51]. For more informations also consult[1, 2, 34].

Again, as earlier, much more remains to be done in order to understand the complexity of the simulation algorithms as well as their efficient implementations.

4 Conclusion

We have described a rich set of computational problems in differential algebra with concrete applications in dynamics and motion-planning problems in robotics, automatic synthesis of control schemes for nonlinear systems and simulation of physical systems with fixed degrees of freedom. There are several related techniques and algorithms for these problems. However, a complete and unifying algorithmic theory is still absent. We highlight many different techniques based on the following approaches: ideal theoretic approach of Ritt, Galois theoretic approach of Kolchin and Singer and group theoretic technique of Lie.

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