Robotics X

B. Mishra

Courant Institute of Mathematical Sciences

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Chapter 1

Continuous Time System

Lecture #1
Feb. 8, 1994

1.1 Continuous Time Systems

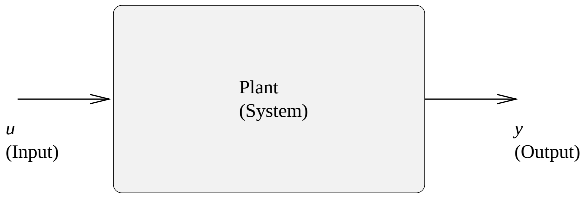

Figure 1.1: Usual model of an input/output system.

$$\begin{array}{cc} \vec{u} \in \mathbb{R}^p & \vec{y} \in \mathbb{R}^q \\ = \text{ Input} & = \text{ output} \end{array}$$

Definition 1.1.1 Static or Zero-Memory System
$\vec{y}(t_0)$ depends only on $\vec{u}(t_0)$. Output depends only on the current input.

Definition 1.1.2 Dynamic System or System with Memory
$\vec{y}(t_0)$ depends on the trajectory of the input $\vec{u}(t) t \in [-\infty, t_0]$.

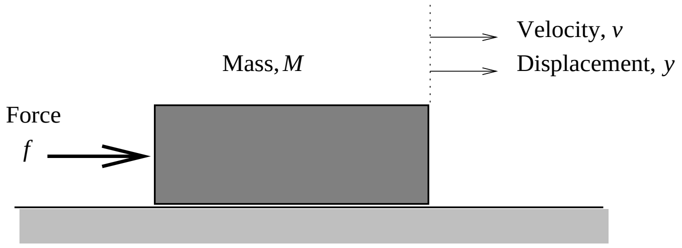

Figure 1.2: A simple dynamical system: a mass moving under a constant force.

Example 1.1.3

$$
\begin{aligned}
f &= M\dot{v} \\
v(t) &= v(t_0) + \frac{1}{M}\int_{t_0}^{t} f(\tau)d\tau \\
y(t) &= y(t_0) + \int_{t_0}^{t} v(\theta)d\theta \\
&= y(t_0) + v(t_0)[t - t_0] \\
&\quad + \frac{1}{M}\int_{t_0}^{t} d\theta \int_{t_0}^{\theta} f(\tau)d\tau
\end{aligned}
$$

Definition 1.1.4 State of the system: *at time $t = t_0$.*
(Phase Space = Position + Momentum)

$$\vec{x}(t) \in \mathbb{R}^n = \textit{State space.}$$

$\diamond\ \vec{x}(t) = \Phi(t, t_0, \vec{x}_0, \vec{u})$
$\equiv$ State Transition Equation.
$\diamond\ \vec{y}(t)\ g(t, x, u)$
$\equiv$ Output Equation

$$
\begin{aligned}
\vec{x}(t_0) &= \vec{x}_0 \\
\dot{\vec{x}}(t) &= f(t, \vec{x}(t), \vec{u}(t)) \\
&\equiv \text{State Evaluation Equation} \\
\vec{y}(t) &= g(t, \vec{x}(t), \vec{u}(t)) \\
&\equiv \text{Output Equation.}
\end{aligned}
$$

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1.2 Discrete-Time System

$$\begin{aligned}\vec{x}(0) &= \vec{x}_0\\ \vec{x}(k+1) &= f(k, \vec{x}(k), \vec{u}(k))\\ \vec{y}(k) &= g(k, \vec{x}(k), \vec{u}(k)).\end{aligned}$$

Time-Invariant Sytem: (Invariant under a time-shift operation.)
Time-Varying System: Otherwise.
Time-Invariant Sytems:

$$\begin{aligned}\dot{\vec{x}}(t) &= f(\vec{x}(t), \vec{u}(t))\\ \vec{y}(t) &= g(\vec{x}(t), \vec{u}(t))\end{aligned}$$

Note: One can write this as

$$\begin{aligned}\dot{\vec{x}}(t) &= f(\vec{x}(t), \vec{u}(t))\\ \vec{y}(t) &= g(\vec{x}(t))\\ &\text{or}\\ \vec{x}(k+1) &= f(\vec{x}(k), \vec{u}(k))\\ \vec{y}(k) &= g(\vec{x}(k)).\end{aligned}$$

◇ **LINEARITY**

f and g are linear maps. (Ie they satisfy principle of superposition.)

◇ **LINEAR TIME-INVARIANT SYSTEMS**

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t)\end{aligned}$$

$$A = n \times n \qquad B = n \times p \quad \text{and} \quad C = q \times n$$

$$\begin{aligned}x(K+1) &= Ax(K) + Bu(K)\\ u(k) &= Cx(K)\end{aligned}$$

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$\diamond$ **Goal:** Generate appropriate inputs so as to constrain the behavior of the dynamical system.

1.3 Set of Possible State-Space Trajectories

$$H_x \stackrel{\Delta}{=} \{h_x\colon\ T \to X\}$$

Desired behavior of the system is given by a *goal set*:

$$\mathcal{G} \subset H_x$$

1.3.1 Servo Problem

Given a reference trajectory

$$g \in H_x$$

$$\mathcal{G} = \{h_x\colon\ |h_x - g| \leq \varepsilon\}$$

Repeat or track the trajectory as closely as possible.

1.3.2 Set-Point Regulation Problem

Given a set of states

$$C \subset X$$

$$\mathcal{G} = \{h_x\colon\ \exists_{t'} \quad \forall_{t>t'} \quad h_x(t) \in C\}$$

Achieve and maintain the particular set of states (C) starting from any initial state.

1.3.3 State-Advoidance Problem

$$Q \subset X = \text{ Set of states}$$

$$G = \{h_x\colon\ \forall_t \quad h_x(t) \notin Q\}$$

Starting from any initial state $\notin Q$, avoid Q.

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1.4 Controllability

An event $\langle \tau, x \rangle \in T \times X$ is said to be controllable with respect to a set of target sets

$$C \subset X$$

if and only if

$$\exists_{t>\tau} \, \exists_{v=\text{control}} \, \langle \tau, x \rangle \longrightarrow_v \{t\colon\ t \geq \tau\} \times C \equiv [\tau, \infty] \times C$$

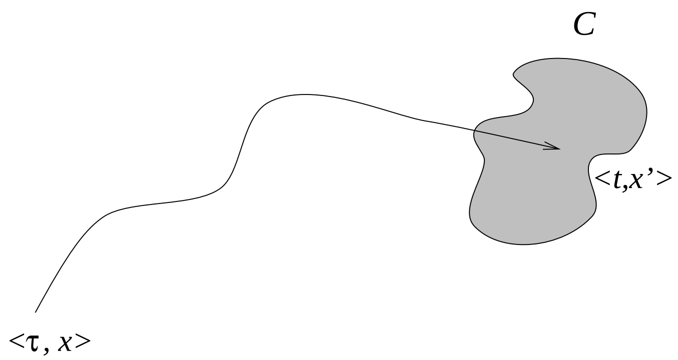

Figure 1.3: What controllability means....

A dynamical system is completely controllable with respect to C if and only if every event in $T \times X$ is controllable with respect to C.

1.4.1 Completely Controllable

A dynamical system is completely controllable if and only if it is possible to transfer any state $x(t_0) \in X$ to any other state in X in finite amount of time $t_1 - t_0$.

1.4.2 Linear Time-Invariant System

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

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Definition 1.4.1 Controllability Matrix

$$[B \mid AB \mid \ldots \mid A^{n-1}B]$$

$$n \times np$$

Theorem 1.4.1 *The system is completely controllable if and only if the rank of the controllability matrix is n. (The rows of the controllability matrix are linearly independent.)*
PROOF.

$$\dot{x}(t) = Ax(t) + Bu(t)$$

The solution is given by

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau.$$

Note:

$$\begin{aligned} e^{At} &= I + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k \\ &= \sum_{i=0}^{\infty} \frac{1}{i!}A^it^i \end{aligned}$$

Facts: 1) $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$
2) $e^{A0} = I$

$$\begin{aligned} x(t) &= e^{At}\left[x_0 + \int_0^t e^{-A\tau}Bu(\tau)d\tau\right] \\ \dot{x}(t) &= Ae^{At}\left[x_0 + \int_0^t e^{-A\tau}Bu(\tau)d\tau\right] + e^{At}[e^{-At}Bu(t)] \\ &= Ax(t) + Bu(t) \end{aligned}$$

(A) The system is controllable **iff** the rows of $e^{-At}B$ are linearly independent **iff**

$$\begin{aligned} W(0,t) &= \int_0^t e^{-A\tau}BB^Te^{-A^T\tau}d\tau \\ &= \text{nonsingular} \qquad \forall t > 0 \end{aligned}$$

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(B) Choose

$$
\begin{aligned}
u(t) &= -B^T e^{-A^T t} W^{-1}(0,t_1)[x_0 - e^{-At_1} x_1] \\
\therefore \quad x(t_1) &= e^{At_1}\left[x_0 + \int_0^{t_1} e^{-A\tau} B(-B^T e^{-A^T\tau} W^{-1}(0,t_1)[x_0 - e^{-At_1}x_1])d\tau\right. \\
&= e^{At_1}\left[x_0 - \int_0^{t_1} e^{-A\tau} BB^T e^{-A^T\tau} d\tau W^{-1}(0,t_1)[x_0 - e^{-At_1}x_1]\right] \\
&= x_1
\end{aligned}
$$

(C) Note that the rows of $e^{-At}B$ are linearly independent $\Leftrightarrow$ $e^{At}B$ satisfies the same condition

$$e^{At}B = B + ABt + \frac{1}{2!}A^2Bt^2 + \cdots$$

has linearly independent rows for all t
$\Leftrightarrow$

$$\therefore \quad [B \mid AB \mid \ldots \mid A^{n-1}B \mid A^nB \mid A^{n+1}B \mid \ldots]$$

has linearly independent rows
$\Leftrightarrow$ The controllability matrix

$$[B \mid AB \mid \ldots \mid A^{n-1}B]$$

has linearly independent rows. [Use Cayley-Hamilton Theorem.]

$$
\begin{aligned}
x(A) &= \text{characteristic equation of } A = \\
&= \det(A - \lambda I) \\
&= \lambda^n + \alpha_1\lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n = 0 \\
\therefore \quad A^n &= -\alpha_1 A^{n-1} - \cdots - \alpha_{n-1}A - \alpha_n I \\
A^nB &= -\alpha_1 A^{n-1}B - \cdots - \alpha_{n-1}AB - \alpha_n B, \text{ etc.}
\end{aligned}
$$

□

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Chapter 2

Controllability and Observability

Lecture #2
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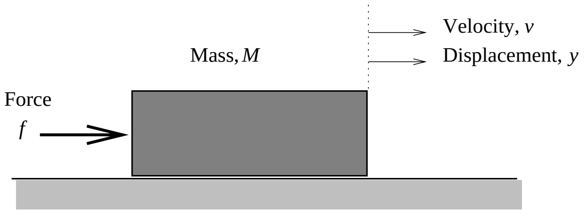
2.1 Controllability

$$\begin{aligned} f &= M\dot{v} \\ v &= \dot{y} \end{aligned}$$



$$\begin{aligned} \begin{bmatrix} \dot{y} \\ \dot{v} \end{bmatrix} &= \begin{bmatrix} v \\ f/M \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} f. \end{aligned}$$

$$B = \begin{bmatrix} 0 \\ 1/M \end{bmatrix}$$

$$AB = \begin{bmatrix} 1/M \\ 0 \end{bmatrix}$$

Thus

$$\text{Controllability matrix} = [B \mid AB] = \begin{bmatrix} 0 & 1/M \\ 1/M & 0 \end{bmatrix} = \text{ of rank 2.}$$

$\therefore$ The system is completely controllable.

2.2 Observability

GOAL: is to recover the state by observing the output trajectory.

Definition 2.2.1 Observer
A system is said to be **completely observable** *if it is possible to identify any state*

$$x(t_0) \in X$$

by observing the output $y(t)$ *for* $t_0 \le t \le t_1$.

Thus "State-determination" or "observation" (also called, "state-reconstruction", a closely related concept) simply means = Identify the state $x(t_1) \in X$ by observing the output $y(t)$ for

$$t_0 \le t \le t_1.$$

Linear-time-invariant system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) \end{aligned}$$

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Definition 2.2.2 Observability Matrix

$$\begin{bmatrix} C \\ \hline CA \\ \hline \vdots \\ \hline CA^{n-1} \end{bmatrix} \qquad nq \times n \quad \textit{matrix.}$$

Theorem 2.2.1 *The system is completely observable iff the rank of its observability matrix is n.*

(The columns of the observability matrix are linearly independent.)

PROOF.

$$\begin{aligned} V(0,t) &= \int_0^t e^{A^T\tau} C^T C e^{A\tau} d\tau \\ &= \text{nonsingular} \quad \forall_{t>0}. \end{aligned}$$

iff

Ce^{At} has linearly independent columns

iff

The system is completely observable.

$$\begin{aligned} \hat{y}(t) &= y(t) - Ce^{At}\int_0^t e^{-A\tau} Bu(\tau) d\tau \\ &\qquad (= y(t),\ \text{if } u(\tau) = 0, \quad 0 \le \tau \le t) \\ &= Cx(t) - Ce^{At}\int_0^t e^{-A\tau} Bu(\tau) d\tau \\ &= Ce^{At}\left[x_0 + \int_0^t e^{-A\tau} Bu(\tau) d\tau\right] \\ &\qquad -Ce^{At}\int_0^t e^{-A\tau} Bu(\tau) d\tau \\ &= Ce^{At} x_0 \end{aligned}$$

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Now,

$$\begin{aligned}
&V^{-1}(0,t_1)\int_0^{t_1} e^{A^T\tau}C^T\hat{y}(\tau)d\tau \\
&= V^{-1}(0,t_1)\int_0^{t_1} e^{A^T\tau}C^TCe^{A\tau}x_0 d\tau \\
&= V^{-1}(0,t_1)V(0,t_1)x_0 = x_0.
\end{aligned}$$

□

2.3 Principle of Duality

The system (Linear Time-invariant) represented by

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{aligned}$$

is controllable at time t_0 its dual system (**adjoint system**)

$$\begin{aligned}
\dot{z}(t) &= -A^Tz(t) + C^Tv(t) \\
w(t) &= B^Tz(t)
\end{aligned}$$

is observable at time t_0.

2.4 Stability

The property that small changes in input or initial conditions do not result in a large change in the system's behavior

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{aligned}$$

$u(t) = u_c =$ constant input.
◇ If there exists a point $x_e \in \mathbb{R}$ s.t.

$$Ax_e + Bu_c = 0$$

then we say that

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- x_e = An equilibrium point of the system corresponding to the input u_c.

◇ **ASSUMPTIONS:**

1. The system has only one equilibrium point.
2. Without loss of generality, we assume that the origin of the state space is such an equilibrium point.
3. $u_c = 0 \sim$ zero-control $\therefore \dot{x}(t) = Ax(t)$.

Definition 2.4.1 *This system is* **stable in the sense of Lyapunov** *at the origin if, for every $\epsilon > 0$,*

$$\begin{aligned} &\exists \delta > 0 \quad \textit{such that} \quad \|x(t_0)\| \leq \delta \\ &\Rightarrow \quad \|x(t)\| \leq \varepsilon \qquad \forall_{t \geq t_0}. \end{aligned}$$

Definition 2.4.2 *The system is asymptotically stable at the origin if*

1. *it is stable in the sense of Lyapunov.*
2. $\exists_{r>0}$ $(r \in \mathbb{R})$

$$\|x(t_0)\| \leq r \Rightarrow x(t) \downarrow 0 \textit{ as } t \uparrow \infty.$$

Theorem 2.4.1 *The system described by the state equation*

$$\dot{x} = Ax(t) + Bu(t)$$

is **asymptotically stable iff** *all of the eigenvalues of the matrix A have negative real parts.*

PROOF.

$$\begin{aligned} \dot{x}(t) &= Ax(t) \\ x(t) &= e^{At}x_0 \end{aligned}$$

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Thus

$$\begin{aligned} A = U\Lambda U^T = U &= \text{Eigenvectors (orthonormal)} \\ \Lambda &= \text{diag}(\lambda_0, \lambda_1, \ldots, \lambda_n) \\ &= \text{eigenvalues.} \end{aligned}$$

$$\begin{aligned} UU^T &= U^T U = I \\ e^{At} &= I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^e t^3 + \cdots \\ &= UU^T + U\Lambda U^T t + \frac{1}{2!} U\Lambda U^T U \Lambda U^T t^2 + \cdots \\ &\quad + \frac{1}{k!} \underbrace{U\Lambda U^T U \Lambda U^T \ldots U\Lambda U^T t^k}_{k} + \cdots \\ &= UU^T + U(\Lambda t)U^T + U\left(\frac{1}{2!}\Lambda^2 t^2\right) U^T + \cdots \\ &\quad + U\left(\frac{1}{k!}\Lambda^k t^k\right) U^T + \cdots \\ &= U(e^{\Lambda t})U^T \\ &= U \begin{bmatrix} e^{\lambda_0 t} & & & \bigcirc \\ & e^{\lambda_1 t} & & \\ \bigcirc & & \ddots & \\ & & & e^{\lambda_{n-1} t} \end{bmatrix} U^T \end{aligned}$$

$\therefore e^{At} \to 0$ as $t \downarrow \infty$ **iff** all of the eigenvalues of the A have negative real parts. $\square$

2.4.1 Companion Forms

Single-Input Systems:

$$\dot{x} = Ax + bu$$

Assume that the system is controllable. That is,

$$\begin{aligned} u &= [b \mid Ab \mid \ldots \mid A^{n-1}b] \quad (n \times n \text{ matrix}) \\ &= \text{of full rank and hence, invertible.} \end{aligned}$$

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Let

$$
\begin{aligned}
p_1 &= [0 \quad 0 \quad \ldots \quad 0 \quad 1]U^{-1} \\
\text{and } P &= \begin{bmatrix} p_1 \\ p_1 A \\ \vdots \\ p_1 A^{n-1} \end{bmatrix} \\
\bar{x} = P_x &= \begin{bmatrix} p_1 \\ p_1 A \\ \vdots \\ p_1 A^{n-1} \end{bmatrix} x
\end{aligned}
$$

$$
\begin{aligned}
\bar{x}_1 &= p_1 x \\
\bar{x}_2 &= p_1 A x \\
\bar{x}_3 &= p_1 A^2 x \\
\bar{x}_n &= p_1 A^{n-1} x
\end{aligned}
$$

$$
\begin{aligned}
p_1 u &= [p_1 b \mid p_1 A b \mid \ldots \mid p_1 A^{n-1} b] \\
&= [0 \quad 0 \quad \ldots \quad 0 \quad 1] u^{-1} u \\
&= [0 \quad \ldots \quad 0 \quad 1] \\
p_1 b &= p_1 A b = \cdots = p_1 A^{n-2} b = 0 \\
p_1 A^{n-1} b &= 1
\end{aligned}
$$

$$
\begin{aligned}
\bar{x}_1 &= p_1 x \Rightarrow \dot{\bar{x}}_1 = p_1 \dot{x} = p_1 A x + p_1 b u = \bar{x}_2 \\
\bar{x}_2 &= p_1 A x \Rightarrow \dot{\bar{x}}_2 = p_1 A \dot{x} = p_1 A^2 x + p_1 A b u = \bar{x}_3 \\
&\vdots \\
\bar{x}_{n-1} &= p_1 A^{n-2} x \Rightarrow \dot{\bar{x}}_{n-1} = p_1 A^{n-2} \dot{x} = p_1 A^{n-1} x + p_1 A^{n-2} b u = \bar{x}_u \\
\bar{x}_n &= p_1 A^{n-1} x \Rightarrow \dot{\bar{x}}_n = p_1 A^{n-1} \dot{x} = p_1 A^n x + p_1 A^{n-1} b u = p_1 A^n x + u
\end{aligned}
$$

But

$$
\begin{aligned}
x(A) &= \det(A - \lambda I) \\
&= \lambda^n + \alpha_{n-1} \lambda^{n-1} + \cdots + \alpha_1 \lambda + \alpha_0 = 0
\end{aligned}
$$

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$\therefore$ By Cayley-Hamilton

$$\begin{array}{rcl} A^n & = & -\alpha_{n-1}A^{n-1} - \cdots - \alpha_1 A - \alpha_0 I \\ p_1 A^n x & = & -\alpha_{n-1}p_1 A^{n-1}x - \cdots - \alpha_1 p_1 A x - \alpha_0 p_1 x \\ & = & -\alpha_0 \bar{x}_1 - \alpha_1 \bar{x}_2 - \cdots - \alpha_{n-1}\bar{x}_n \\ \therefore \quad \dot{\bar{x}}_n & = & -\alpha_0 \bar{x}_1 - \alpha_1 \bar{x}_2 - \cdots - \alpha_{n-1}\bar{x}_n + u \end{array}$$

In the matrix form:

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \vdots \\ \dot{\bar{x}}_{n-1} \\ \dot{\bar{x}}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \ldots & -\alpha_{n-1} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_{n-1} \\ \bar{x}_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

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Chapter 3

Modal Control

Lecture #3
Feb. 22, 1994

3.1 Modal Control

(State-Feedback Control System)

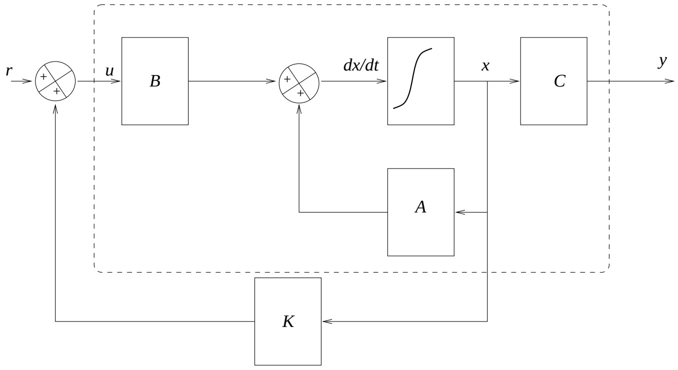
Goals:

1. State-Feedback Regulation Problem (Design to be stable)
   (a) Limit cycles must shrink
   (b) Non-chaotic system etc.
2. Servo Problem
   (a) Output $y(t)$ should approximate the desired value within a prescribed tolerance.
   (b) Output should approach its desired value rapidly and without excessive overshoot.
   (c) Decoupling: One input affects one output, etc.
   (d) Insensitive to parameter variation.

(e) Immune to noise.

3. Structure of the controller must be simple.

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx \\
u &= r + Kx \\
&\Downarrow \\
\dot{x} &= (A + Bk)x + Br \\
y &= Cx
\end{aligned}
$$



What is the effect of the control law?
Single-input System (in companion form)

$$
\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \vdots \\ \dot{\bar{x}}_{n-1} \\ \dot{\bar{x}}_n \end{bmatrix}
=
\underbrace{\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \dots & -\alpha_{n-1} \end{bmatrix}}_{A}
\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_{n-1} \\ \bar{x}_n \end{bmatrix}
+
\underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{b} u
$$

choose

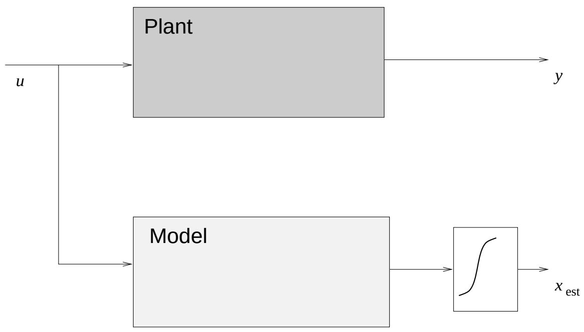
$$K = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \alpha_0 - \bar{\alpha}_0 & \alpha_1 - \bar{\alpha}_1 & \alpha_2 - \bar{\alpha}_2 & \dots & \alpha_{n-1} - \bar{\alpha}_{n-1} \end{bmatrix}$$

$$A + bK = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \bar{\alpha}_0 & -\bar{\alpha}_1 & -\bar{\alpha}_2 & \dots & \bar{\alpha}_{n-1} \end{bmatrix}$$

Pole Placement: Note: Characteristic Equation and hence the eigenvalues can be modified as desired.

3.2 Problem with State-Feedback Control

3.2.1 Model-Based Control



$$\text{System (Plant)}: \quad \dot{x} = Ax + Bu$$
$$y = Cx$$

$$\text{Model}: \quad \dot{\hat{x}} = \tilde{A}\hat{x} + \widetilde{B}u$$

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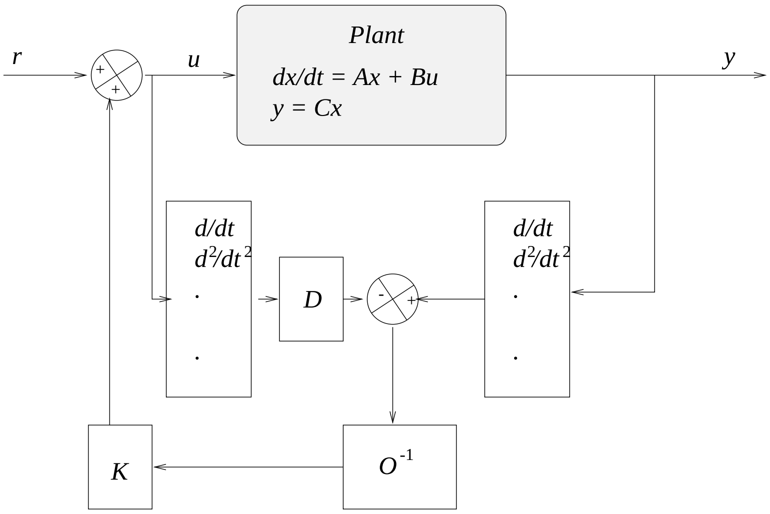
1. If $\tilde{A} \neq A$ and $\widetilde{B} \neq B$ then the estimated state may be highly erroneous.

2. Initial state value may be wrong.

3. If A has positive eigenvalues (i.e. its Lyapunov coefficient > 0) then the system may be inherently chaotic, and

$$\hat{x} \neq x. \text{ (In the presence of noise.)}$$

3.2.2 State-Observation by Differentiation



$$
\begin{aligned}
y &= Cx \\
\dot{y} &= C\dot{x} = CAx + CBu \\
\ddot{y} &= CA\dot{x} + CB\dot{u} = CA^2x + CABu + CB\dot{u} \\
\dddot{y} &= CA^2\dot{x} + CAB\dot{u} + CB\ddot{u} \\
&= CA^3x + CA^2Bu + CAB\dot{u} + CB\ddot{u} \\
&\vdots \\
y^{(n-1)} &= CA^{n-1}x + CA^{n-2}Bu + \cdots + CABu^{(n-3)} + CBu^{(n-2)}
\end{aligned}
$$

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$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} x = \underbrace{\begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix} - \underset{n \times n-1}{\begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ CB & 0 & \dots & 0 & 0 \\ CAB & CB & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ CA^{n-2}B & CA^{n-3}B & \dots & CAB & CB \end{bmatrix}} \begin{bmatrix} u \\ \dot{u} \\ \ddot{u} \\ \vdots \\ u^{n-2} \end{bmatrix}}_{f(Y,U)}$$

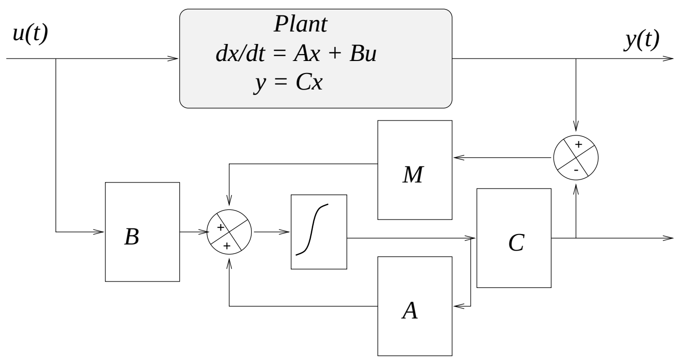
$$x = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^{-1} f(Y,U)$$

Solvable, if the system is observable. The observability matrix is of full rank.

Problem: If there is noise in the system then the differentiation creates high frequency noise in the state estimation.

3.3 Luenberger Observer

(Assuming that the system is observable.)



$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu - M\widetilde{y} \\ &= A\hat{x} + Bu - M(y - \hat{y}) \\ &= A\hat{x} + Bu - MCx + MC\hat{x} \end{aligned}$$

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$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
(\dot{\hat{x}} - \dot{x}) &= A(\hat{x} - x) + MC(\hat{x} - x) \\
\widetilde{x} &= \hat{x} - x \\
\therefore \quad \dot{\widetilde{x}} &= (A + MC)\widetilde{x} \\
\therefore \quad \widetilde{x} &= e^{(A+MC)t}\widetilde{x}(0)
\end{aligned}
$$

If the system is observable, then M can be so chosen that all eigenvalues of $A+MC$ have $-ve$ real parts $\Rightarrow$ **Asymptotically Stable Observer**

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Chapter 4

Discrete Event System

Lecture #4
Feb. 22, 1994

4.1 Discrete Event Systems (DES)

Dynamical systems that can be represented in terms of **finite state automata:**

Definition 4.1.1 DES (Discrete Event System) *is represented as an automaton*

$$G = (U, X, f, x_0)$$

where

$$\begin{aligned} U &= \textit{Set of inputs/controls} \\ &= \textit{Set of primitive events} \\ X &= \textit{Set of states} \\ f &: U \times X \to X = \textit{ The transition functions} \\ x_0 &= \textit{initial state} \end{aligned}$$

Partition U into two subsets

$$U_c = \text{Set of controllable events}$$

$$\begin{aligned} U_u &= \overline{U}_c = U - U_c = \\ &= \text{Set of uncontrollable events} \end{aligned}$$

An **admissible control** for such a dynamical system consists of a subset $r \subseteq u$ such that

$$U_u \subseteq r \subseteq u$$

$r =$ **set of enabled events**
Note: All the uncontrollable events are always enabled.

$$\begin{aligned} \Gamma &= 2^u \\ \Gamma &= \text{Set of all } \textbf{admissible controls} \end{aligned}$$

$r \in \Gamma$ and $u \in r$

$$\Rightarrow \quad u = \text{ enabled by } r$$
$$\text{otherwise } u = \text{disabled by } r.$$

Control consists of disabling some controllable events, depending only upon the current state.

4.2 Modal Control

A controller for a given dynamical system is specified as a map

$$\eta\colon\ X \to \Gamma$$

1. Disabled events are prevented from occurring.

2. Enabled events are allowed to occur if permitted by the underlying dynamics.

$$\forall_{r \in \Gamma} \qquad U_u \subseteq r$$

The controller cannot prevent an uncontrolled event from happening (if the dynamics permits it).

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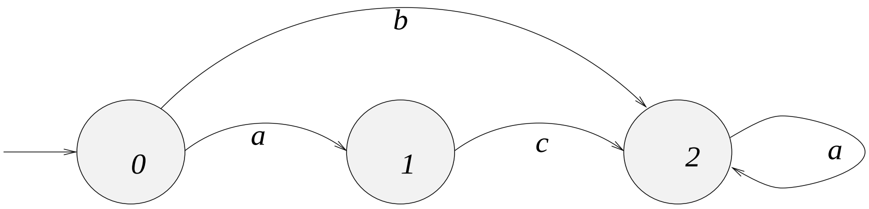
Definition 4.2.1 Liveness Condition

$$\forall_{x \in X} \quad \exists_{u \in \eta(x)} \quad \exists_{y \in X} \quad f(u,x) = y$$

(No deadlock.)

The controller ensures that for any state there is at least one enabled event for which the transition function is defined. The system can remain in the same state only if it is permitted to do so dictated by the dynamics.



Example 4.2.2

$$u = \{a, b, c\} \qquad x = \{0, 1, 2\}$$

$$x_0 = 0$$

$$\begin{aligned} f &: u \times X \longrightarrow X \\ &: (a, 0) \longmapsto 1 \\ &: (b, 0) \longmapsto 2 \\ &: (c, 1) \longmapsto 2 \\ &: (a, 2) \longmapsto 2 \end{aligned}$$

$$\begin{aligned} U_c &= \{a, b\} \\ U_u &= \{c\} \end{aligned}$$

Controller

$$\begin{aligned} \eta &: X \longrightarrow \Gamma \\ &: 0 \longmapsto \{b, c\} \\ &: 1 \longmapsto \{c\} \\ &: 2 \longmapsto \{a, c\} \end{aligned}$$

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CDES (Controlled Discrete Event System) achieves the state $\{2\}$ *while avoiding the state* $\{1\}$.

$$\begin{aligned} FSA &= \textit{Generator for a language} \\ u^* &= \textit{Kleene Closure (Kleene Star)} \\ &= \textit{Set of all finite-length strings} \\ & \textit{over the set of primitive events } u. \end{aligned}$$

$$L \subseteq U^*, L = \textit{ language generated by the FSA.}$$

$$L = ba^* + aca^*.$$

The controller modifies the language to

$$\underline{L' = ba^*}.$$

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Chapter 5

Digression

Lecture #5
Mar. 1, 1994

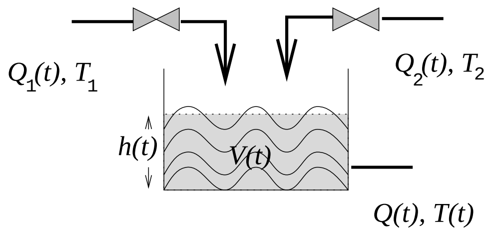
5.1 Digression

Example 5.1.1

$$\begin{aligned}
Q(t) &= k\sqrt{h(t)} = k\sqrt{\frac{V(t)}{A}} \\
\therefore \quad \frac{\partial Q(t)}{\partial V(t)} &= \frac{k}{2\sqrt{A(V(t)}} \\
&= \frac{k}{2}\sqrt{\frac{V(t)}{Q}} \cdot \frac{1}{V(t)} \\
&= \frac{1}{2}\frac{Q(t)}{V(t)}
\end{aligned}$$

1. Equation for the rate of change of the mass:

$$\frac{dV(t)}{dt} = Q_1(t) + Q_2(t) - Q(t)$$



2. Equation for the rate of change of the energy:

$$\frac{d}{dt}[\rho C_p T(t)V(t)] = \rho C_p T_1 Q_1(t) + \rho C_p T_2 Q_2(t) - \rho C_p T(t)Q(t)$$

where

$$\left.\begin{array}{rcl} \rho & = & \text{density} \\ C_p & = & \text{Specific Heat} \end{array}\right\} \quad \text{Constants}$$

$$\therefore \quad \frac{d}{dt}[T(t)V(t)] = T_1 Q_1(t) + T_2 Q_2(t) - T(t)Q(t)$$

Writing down the quantities about their equilibrium values:

$$\begin{array}{rcl} Q(t) & = & Q_0 + \widetilde{Q}(t) \\ Q_1(t) & = & Q_{10} + \widetilde{Q}_1(t) \\ Q_2(t) & = & Q_{20} + \widetilde{Q}_2(t) \\ T(t) & = & T_0 + \widetilde{T}(t) \\ V(t) & = & V_0 + \widetilde{V}(t) \end{array}$$

where Q_0, Q_{10}, Q_{20}, T_0 and V_0 denote the equilibrium values.

Let $V/_0/Q_0 \equiv \tau$

Hence:

$$\begin{array}{l} Q_{10} + Q_{20} - Q_0 = 0 \\ \text{and} \qquad T_1 Q_{10} + T_2 Q_{20} - T_0 Q_0 = 0 \end{array}$$

$$Q(t) \quad = \quad Q_0 + \frac{\partial Q(t)}{\partial V(t)}\Big|_{V(t)=V_0} (V(t) - V_0)$$

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$$
\begin{aligned}
&= Q_0 + \frac{1}{2}\frac{Q_0}{V_0}\widetilde{V}(t) \\
\therefore \quad \widetilde{Q}(t) &= \frac{1}{2\tau}\widetilde{V}(t) \\
\therefore \dot{\widetilde{V}}(t) &= \widetilde{Q}_1(t) + \widetilde{Q}_2(t) - \widetilde{Q}(t) \\
&= \widetilde{Q}_1(t) + \widetilde{Q}_2(t) - \widetilde{V}(t)/2\tau
\end{aligned}
$$

$$
\begin{aligned}
&\dot{\widetilde{T}}(t)V_0 + T_0\dot{\widetilde{V}}(t) \\
&\quad = T_1\widetilde{Q}_1(t) + T_2\widetilde{Q}_2(t) - T_0\widetilde{Q}(t) - Q_0\widetilde{T}(t) \\
&\quad = T_1\widetilde{Q}_1(t) + T_2\widetilde{Q}_2(t) - T_0\widetilde{V}(t)/2\tau - V_0\widetilde{T}(t)/\tau \\
\text{L.H.S.} &\quad = \dot{\widetilde{T}}(t)V_0 + T_0\widetilde{Q}_1(t) + T_0\widetilde{Q}_2(t) - T_0\widetilde{V}(t)/2\tau
\end{aligned}
$$

(Substitute the equation for $\dot{\widetilde{V}}(t)$)

$\therefore$

$$
\dot{\widetilde{T}}(t)V_0 = (T_1 - T_0)\widetilde{Q}_1(t) + (T_2 - T_0)\widetilde{Q}_2(t) - \frac{V_0}{\tau}\widetilde{T}(t)
$$

Hence, we have

$$
\begin{aligned}
\dot{\widetilde{V}}(t) &= -\frac{1}{2\tau}\widetilde{V}(t) + \widetilde{Q}_1(t) + \widetilde{Q}_2(t) \\
\dot{\widetilde{T}}(t) &= -\frac{1}{\tau}\widetilde{T}(t) + \left(\frac{T_1 - T_0}{V_0}\right)\widetilde{Q}_1(t) \\
&\quad + \left(\frac{T_2 - T_0}{V_0}\right)\widetilde{Q}_2(t) \\
\begin{bmatrix} \dot{\widetilde{V}}(t) \\ \dot{\widetilde{T}}(t) \end{bmatrix} &= \begin{bmatrix} -1/2\tau & 0 \\ 0 & -1/\tau \end{bmatrix} \begin{bmatrix} \widetilde{V}(t) \\ \widetilde{T}(t) \end{bmatrix} \\
&\quad + \begin{bmatrix} 1 & 1 \\ \frac{T_1 - T_0}{V_0} & \frac{T_2 - T_0}{V_0} \end{bmatrix} \begin{bmatrix} \widetilde{Q}_1(t) \\ \widetilde{Q}_2(t) \end{bmatrix} \\
\begin{bmatrix} \widetilde{Q}(t) \\ \widetilde{T}(t) \end{bmatrix} &= \begin{bmatrix} 1/2\tau & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{V}(t) \\ \widetilde{T}(t) \end{bmatrix}
\end{aligned}
$$

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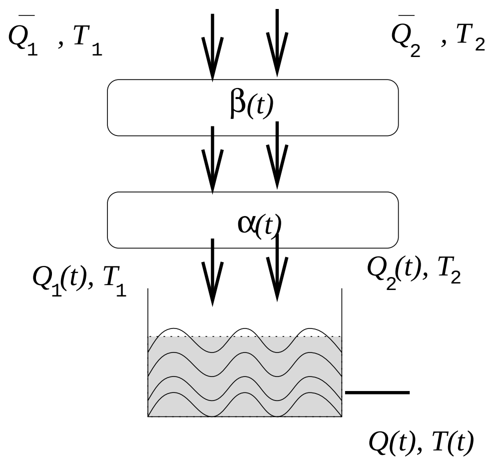
5.1.1 A Variation

Example 5.1.2

$$
\begin{aligned}
\therefore \quad \frac{d}{dt}V(t) &= Q_1(t) + Q_2(t) - Q(t) \\
&= \alpha(t)\overline{Q} - Q(t) \\
\frac{d}{dt}[T(t)V(t)] &= T_1Q_1(t) + T_2Q_2(t) - T(t)Q(t) \\
&= \alpha(t)T_2\overline{Q} + \alpha(t)\beta(t)(T_1 - T_2)\overline{Q} - T(t)Q(t)
\end{aligned}
$$



Let

$$
\begin{aligned}
Q(t) &= Q_0 + \widetilde{Q}(t) \\
T(t) &= T_0 + \widetilde{T}(t) \\
V(t) &= V_0 + \widetilde{V}(t) \\
\alpha(t) &= \alpha_0 + \tilde{\alpha}(t) \\
\beta(t) &= \beta_0 + \tilde{\beta}(t)
\end{aligned}
$$

(1) $$\alpha_0\overline{Q} - Q_0 = 0 \quad \Rightarrow \overline{Q} = Q_0/\alpha_0$$

(2) $$\alpha_0T_2\overline{Q} + \alpha_0\beta_0(T_1 - T_2)\overline{Q} - T_0Q_0 = 0$$

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(3)
$$Q_0 = K\sqrt{\frac{V_0}{A}}$$
$$\& \widetilde{Q}(t) = \widetilde{V}(t)/2\tau$$
$$\dot{\widetilde{V}}(t) = \tilde{\alpha}(t)\frac{Q_0}{\alpha_0} - \frac{1}{2\tau}\widetilde{V}(t)$$

$$\begin{aligned}
\dot{\widetilde{T}}(t)V_0 + T_0\widetilde{V}(t) &= \dot{\widetilde{T}}(t)V_0 + T_0Q_0 \\
&= T_0\frac{Q_0}{\alpha_0}\tilde{\alpha}(t) + (T_1 - T_2)Q_0\tilde{\beta}(t) \\
&\quad +(T_1 - T_2)\frac{Q_0\beta_0}{\alpha_0}\tilde{\alpha}(t) - \widetilde{T}(t)Q_0 \\
&= T_2\overline{Q}\alpha_0 + (T_1 - T_2)\alpha_0\beta_0\left(\frac{\tilde{\alpha}(t)}{\alpha_0}\right) \\
&\quad +(T_1 - T_2)Q_0\tilde{\beta}(t) - \widetilde{T}(t)Q_0 \\
\therefore \quad \dot{\widetilde{T}}(t)V_0 &= (T_1 - T_2)Q_0\tilde{\beta}(t) - \widetilde{T}(t)Q_0 \\
\therefore \quad \dot{\widetilde{V}}(t) &= -1/2\tau\widetilde{V}(t) + \overline{Q}\tilde{\alpha}(t) \\
\dot{\widetilde{T}}(t) &= -1/\tau\widetilde{T}(t) + \frac{T_1 - T_2}{\tau}\tilde{\beta}(t) \\
\begin{bmatrix} \dot{\widetilde{V}}(t) \\ \dot{\widetilde{T}}(t) \end{bmatrix} &= \begin{bmatrix} -1/2\tau & 0 \\ 0 & -1/\tau \end{bmatrix}\begin{bmatrix} \widetilde{V}(t) \\ \widetilde{T}(t) \end{bmatrix} + \begin{bmatrix} \overline{Q} & 0 \\ 0 & \frac{T_1 - T_2}{\tau} \end{bmatrix}\begin{bmatrix} \tilde{\alpha}(t) \\ \tilde{\beta}(t) \end{bmatrix} \\
\begin{bmatrix} \widetilde{Q}(t) \\ \widetilde{T}(t) \end{bmatrix} &= \begin{bmatrix} 1/2\tau & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \widetilde{V}(t) \\ \widetilde{T}(t) \end{bmatrix} \\
\therefore \quad \dot{\widetilde{V}}(t) &= -1/2\tau\widetilde{V}(t) + \overline{Q}\tilde{\alpha}(t) \;\; and \;\; \widetilde{Q}(t) = \frac{\widetilde{V}(t)}{2\tau} \\
\dot{\widetilde{T}}(t) &= -1/\tau\widetilde{T}(t) + T_1 - T_2/\tau\tilde{\beta}(t)
\end{aligned}$$

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Chapter 6

Controlled DES

Lecture #6
Mar. 1, 1994

6.1 Controlled Discrete Event Systems (CDES)

Definition 6.1.1 Supervisor *for a discrete event system (DES)*

$$\begin{aligned} \eta \ &: \ L \longrightarrow \Gamma \\ &: \ w \longmapsto \eta(w) = r \in \Gamma \subseteq 2^u \end{aligned}$$

wher

L = *Language (behavior) generated by the discrete event system*
Γ = *Set of all admissible controls.*

◇ Our Objective:
Design a supervisor that selects control inputs in such a way that the given CDES (Controlled Discrete Event System) behaves in obedience to various constraints.
Constraints

1. Certain undesirable sequences of events are not permitted to occur.

2. Certain other desirable events are permitted to occur.

$$G = \text{CDES}, \qquad \eta = \text{Supervisor}$$

A "Liveness-Like" condition:

Following the generation of a string w, the next event must be an element of

$$\eta(w) \cap u(f(w, q_0)) \neq \emptyset.$$

$$\begin{aligned} (G,\eta) &= \text{The closed loop system of } G \text{ supervised by } \eta \\ L_\eta = L(G,\eta) &= \text{Language accepted by the CDES } (G,\eta) \end{aligned}$$

$L_\eta \triangleq (i)\varepsilon \in L_\eta$ and
$(ii) wu \in L_\eta$ <u>iff</u>

$$w \in L_\eta, u \in \eta(w) \quad \text{and} \quad wu \in L(G)$$

$\therefore\ L_\eta \subseteq L(G)$

$$w_1 w_2 \ldots w_n \in L_\eta \Rightarrow \varepsilon \in L_\eta, \quad w_1 \in L_\eta, \ldots, w_1 \ldots w_n \in L_\eta$$

$$w_j \in \eta(w_1 w_2 \ldots w_{j-1})$$

$\therefore\ L_\eta =$ Prefix-closed subset of $L(G)$.

Definition 6.1.2 *The <u>prefix-closure</u> of $L \subseteq U^*$ is that subset $\bar{L} \subseteq U^*$ such that*

$$\bar{L} = \{u\colon\ uv \in L \quad \textit{for some} \quad v \in U^*\}$$

$$u \in \bar{L} \Leftrightarrow \exists_{v \in u^*} \quad uv \in L$$

$$\bar{L} = \textit{Set of all prefixes of } L.$$

$L =$ *prefix-closed (or simply closed) iff* $\bar{L} = L$.

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Supervisor Concrete representation

$$\begin{array}{rcl} G & = & (U, X, f, x_0) \\ T & = & (U, Y, g, y_0) \\ \eta' & : & Y \to \Gamma \end{array}$$

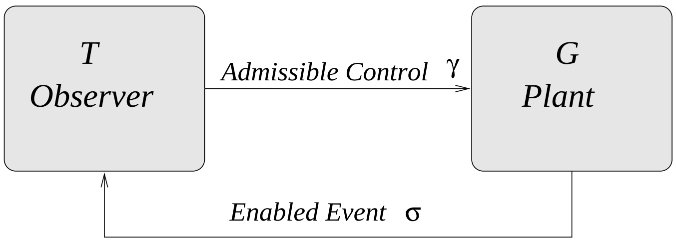
The pair (T, η') realizes the supervisor η if

$$\forall_{w \in L_\eta} \qquad \eta'(f(w, y_0)) = \eta(w)$$

The value of η on the string w can be found by first applying w to T causing T to be driven from its initial state to some state y and then computing $\eta'(y)$.

In control terminology:

$$\begin{array}{rcl} G & = & \text{“plant”} \\ T & = & \text{“Observer” or} \\ & & \text{“Dynamic Compensator”} \\ \eta' & = & \text{“Feedback”} \end{array}$$



Definition 6.1.3 *A language $K \subseteq L$ is said to be* **controllable** *with respect to a given discrete event system if*

$$\overline{K} U_u \cap L \subseteq \overline{K}$$

$\overline{K} U_u$ = Concatenation of a prefix of K with an uncontrollable event in U_u.
$\overline{K}$ = Conditionally invariant under the action of U_u.

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Theorem 6.1.1 *For any discrete event system G with closed behavior L and any regular sublanguage $K \subseteq L$, there exists a supervisor that serves to restrict G to exactly K if and only if*

1. $\overline{K} = K$ *(K = prefix closed) and*
2. $\overline{K}U_u \cap L \subseteq \overline{K}$ *(K = controllable).*

PROOF.

$$\begin{aligned} G &= (U, X, f, x_0) = \text{Plant},\ L = L(G) \\ & \left.\begin{aligned} S &= (T, \eta) \\ &= (U, Y, g, y_0), L(T) = K \\ \eta&\colon\ Y \to \Gamma \text{ (to be specified)} \end{aligned}\right\} \quad \text{Supervisor} \end{aligned}$$

Let

$$\begin{aligned} U_y^0 &= \{u\colon\ (\exists w \in \overline{K}) \quad g(w, y_0) = y, wu \in L, wu \notin K\} \\ U_y^1 &= \{u\colon\ (\exists w \in \overline{K}) \quad g(w, y_0) = y, wu \in L, wu \in K\} \end{aligned}$$

$$\begin{aligned} \eta &: \quad Y \to \Gamma \\ &: \quad y \mapsto U_u \cup U_y^1 \end{aligned}$$

1. $L(G,T) \subseteq L(T) = K$
2. We need to show that $K \subseteq L(G,T)$. Every string in K also belongs to $L(G,T)$.

The proof is by induction on the length of the string

1. $\varepsilon \in K$ and $\varepsilon \in L(G,T)$
2. Let $wu \in K$. By prefix closure $w \in K$ and by inductive hypothesis $w \in L(G,T)$

$$u \in \eta(g(w, y_0) \Rightarrow u \in U_u \text{ or } u \in U_y^1$$

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(a) **Case 1.**

$$u \in U_u \Rightarrow \begin{array}{ll} (i) & wu \in KU_u \\ (ii) & wu \in K \subseteq L \\ & \quad \Rightarrow wu \in L(G,T) \end{array}$$

(b) **Case 2.**

$$u \in U_y^1 \Rightarrow \begin{array}{ll} (i) & wu \in L, \\ (ii) & wu \in K \\ & \quad \Rightarrow wu \in L(G,T) \end{array}$$

□

<u>Remark</u>:

$$G = \text{Plant},\ L = L(G)$$

$$\eta\colon\ L \to \Gamma \quad (=\ \text{Supervisor})$$

We note that it is possible to realize the supervisor η as another DES, T, if the following conditions hold

1. $w \in L_\eta = L(G,\eta)$

$$\Rightarrow w \in L(T) \text{ and } wu \in L(T) \longrightarrow u \in \eta(w)$$

2. $w \in L_\eta \wedge wu \in L \wedge u \in \eta(w)$

$$\Rightarrow wu \in L(T)$$

<u>Thus</u>

- (1) $\Leftrightarrow$ Those transitions disabled by η do not appear in the transition structure of T.
- (2) $\Leftrightarrow$ Those transitions enabled by η and which are possible in G do appear in the transition structure of S.

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Chapter 7

Supremal Controllable Language

Lecture #7
Mar. 8, 1994

7.1 Supremal Controllable Sublanguage

$$\begin{aligned}
U &= \text{Alphabet.} \\
U^* &= \text{Set of all finite strings over } U. \\
M &= \text{A fixed language over } U. \\
U_u &= \text{Uncontrollable events in } U. \\
U &= U_c \uplus U_u
\end{aligned}$$

Recall: $K \subseteq U^* =$ controllable if and only if

$$\mathbf{\overline{K} U_u \cap M \subset \overline{K}}$$

7.1.1 Structure of the Family of Controllable Sublanguages

$\diamond$ <u>Given</u> $L \subseteq M$

$$\begin{aligned} C(L) &= \text{Family of controllable sublanguages of } L \\ L^{\uparrow} &= \sup C(L) \\ &= \cup\{K\colon\ K \subset L \text{ and } K = \text{Controllable}\} \\ &= \text{Supremal (under the order defined by set-inclusion)} \\ &\quad\ \text{controllable sublanguage of } L. \end{aligned}$$

NOTE $C(L)$ and $\sup C(L)$ are well-defined.

1. $C(L)$ is nonempty, since $\phi \subseteq C(L)$

   $$[\bar{\phi} U_u \cap M = \phi \subseteq \bar{\phi} = \phi]$$

   is always controllable.

2. $C(L) =$ closed under set union:

   $$\begin{aligned} & K_1, K_2 \in C(L) \\ &\quad \Rightarrow \quad \overline{K}_1 U_u \cap M \subseteq \overline{K}_1 \text{ and } \overline{K}_2 U_u \cap M \subseteq \overline{K}_2 \\ &\quad \Rightarrow \quad (\overline{K}_1 U_u \cap M) \cup (\overline{K}_2 U_u \cap M) \subseteq \overline{K}_1 \cup \overline{K}_2 \\ &\qquad\qquad\quad = (\overline{K_1 \cup K_2}). \end{aligned}$$

   ($\because$ Union of the prefixes is same as prefixes of the union. <u>Note</u> $\overline{K}_1 \cap \overline{K}_2 \supset \overline{K_1 \cap K_2}$)

   $$\left\{ \begin{array}{lcl} K_1 & = & \{a\}, \\ K_2 & = & \{aaa\} \\ \overline{K}_1 \cap \overline{K}_2 & = & \{\epsilon, a\} \\ \overline{K_1 \cap K_2} = \bar{\phi} & = & \phi \end{array} \right.$$

   $$\begin{aligned} \Rightarrow \quad & (\overline{K}_1 U_u \cup \overline{K}_2 U_u) \cap M = (\overline{K}_1 \cup \overline{K}_2) U_u \cap M \\ & \qquad = (\overline{K_1 \cup K_2}) U_u \cap M \subseteq (\overline{K_1 \cup K_2}) \\ \Rightarrow \quad & K_1 \cup K_2 \in C(L) \\ & K_1, K_2, \ldots, K_j \in C(L) = (\cup K_i) \in C(L) \\ & \therefore \sup C(L) = \bigcup_{K \in C(L)} K \end{aligned}$$

   Hence $C(L)$ has a unique supremal member under the partial ordering of subset inclusion.

   $$\mathbf{L}^{\uparrow} = \sup \mathbf{C}(\mathbf{L})$$

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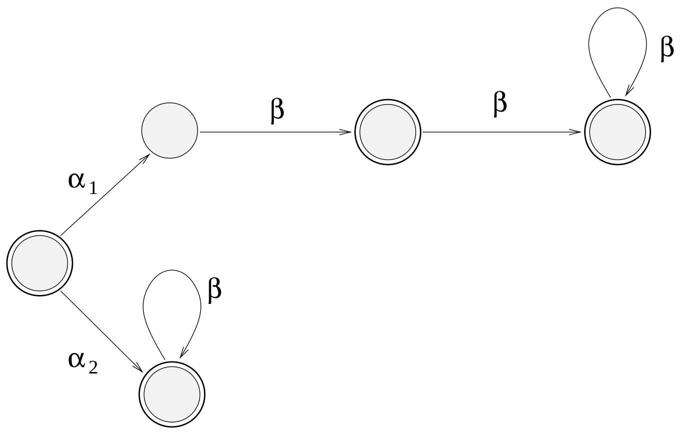

Example 7.1.1

$$\begin{array}{rcl} M & = & \overline{(\alpha, \beta^2 + \alpha_2)\beta^*} \\ U & = & \{\alpha_1, \alpha_2, \beta\} \\ U_c & = & \{\alpha_1, \alpha_2\} \quad U_u = \{\beta\} \end{array}$$

Let $L = \{\alpha, \beta^2\} \subseteq M$

$$L \neq \text{countrollable} \quad (\alpha, \beta^2)\beta \cap M = \alpha, \beta^2 \notin \bar{L}$$

$$\therefore \quad \bar{L}\beta \cap M \not\subseteq \bar{L}$$

and $L^{\uparrow} = \phi$.

Intuitively, in order to try to generate a subset of L you must disable α_2. But, once α_1 happens, we have no control over the β's.

$$L \subseteq M \quad \text{(as before)}$$

$D(L)$ = Family of prefix-closed controllable superlanguages of L (with respect to M).
Note: $D(L)$ = Nonempty and Closed under intersection.

$$\begin{array}{rcl} L^{\uparrow} & = & \inf D(L) = \bigcap_{K \in D(L)} K \\ & = & \text{Uniquely defined} \end{array}$$

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Natural controllable approximation to $L = L^{\uparrow}$.

Preserves the restrictions imposed by L and simultaneously, requires the least amount of control actions. Let $L, M \subset U^*$ be two fixed languages (as before).

$$\begin{aligned} U_u &\subseteq U \\ \mathcal{L} &= \text{ Set of all languages over } U. \end{aligned}$$

Operator:

$$\begin{aligned} \Omega &: \mathcal{L} \longrightarrow \mathcal{L} \\ &: K \longmapsto \\ &\qquad L \cap \sup\{T : T \subseteq U^*, \\ &\qquad\qquad (*) T = \overline{T} \text{ and} T U_u \cap M \subset \overline{K}\} \end{aligned}$$

Note if $K =$ controllable sublanguage of L then

$$T = \overline{K} \quad \text{and} \quad L \cap \overline{K} = \Omega(K)$$

Also note the condition $(*)$ is closed under arbitrary union and hence the supremum under the partial ordering of subset inclusion is well-defined.

$\Rightarrow$ $\Omega =$ well defined and

if $\Omega(K) = K$ then $K =$ controllable sublanguage of L.

Ω depends on L, M and U_u.

$$\Omega(K) = \{t\colon\ t \in L \quad \text{and} \quad \bar{t} U_u \cap M \subset \overline{K}\}$$

$\diamond$ Let us now define the following operators:

$$\begin{aligned} (1) \quad \Gamma &: \mathcal{L} \longrightarrow \mathcal{L} \\ &: K \longmapsto \sup\{T\colon\ T \subset K \text{ and } T = \overline{T}\} \\ (2) \quad X_u &: \mathcal{L} \longrightarrow \mathcal{L} \\ &: K \longmapsto \{w\colon\ w \in U^* \text{ and } wu \in K,\ \forall u \in U_u\} \end{aligned}$$

Notation

$$\begin{aligned} K \quad \Rightarrow \quad & M(K) = \text{Machine accepting} \\ & \|K\| = \#\text{ minimal } \#\text{ of sta of an FSA accepting } K. \end{aligned}$$

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7.1.2 Construction of Γ

$$M(K) = (u, f, x_0, X_f)$$

(1)

$$\begin{array}{rcl} x_0 \notin X_f & \Rightarrow & \Gamma(K) = \phi \\ & & \quad \text{since } \epsilon \notin K \text{ and } K \text{ has no prefix closed subset} \\ & & T \subseteq K \text{ and } T = \overline{T} \text{ and } T \neq \overline{K} \\ & & \quad \rightarrow \varepsilon \in T \subseteq K \\ & \Rightarrow & M(\Gamma(K)) = (U, X, f, x_0, \phi) \end{array}$$

(2) $x_0 \in X_f \Rightarrow$ construct new FSA with states =

$$\begin{array}{rcl} Y & = & Y_f = X_f \\ y_0 & = & x_0 \end{array}$$

and transition function =

$$\begin{array}{rcl} g & : & U \times Y \rightarrow Y \\ & : & (u, y) \mapsto f(u, y) \text{ iff } f(u, y) \in X_f. \end{array}$$

$$M(\Gamma(K)) = (U, Y, g, y_0, Y_f)$$

7.1.3 Construction of X_u

$$\begin{array}{rcl} M(K) & = & (U, X, f, x_0, X_f) \\ Y & = & X, \quad y_0 = x_0 \quad g \equiv f \\ Y_f & = & \{x \in X \colon \forall_{u \in U_u} \quad f(u, x) \in X_f \text{ is } f(u, x) = \text{ undefined}\} \\ M(\Gamma(K)) & = & (U, Y, g, y_0, Y_f) \end{array}$$

7.1.4 Construction of Ω

$$\Omega(K) = L \cap \Gamma \cdot X_u(\overline{K} \cup M^c)$$

Note

$$\Omega(K) \quad = \quad \{t \colon\ t \in L \text{ and } \bar{t}U_u \cap M \subset \overline{K}\}$$

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$$
\begin{aligned}
&= \{t\colon\ t \in L \text{ and } \bar{t}U_u \subset \overline{K} \cup M^c\} \\
&= \{t\colon\ t \in L \text{ and } \bar{t} \in X_u(\overline{K} \cup M^c)\} \\
&= L \cap \Gamma \cdot X_u(\overline{K} \cup M^c)
\end{aligned}
$$

Also, Note $= \Omega =$ polynomially (in $\|L\|, \|K\|$ and $\|M\|$) computable.

Theorem 7.1.1

$$
\begin{aligned}
L^{\uparrow} &= \textit{Largest sublanguage } J \textit{ of } L \textit{ s.t. } \Omega(J) = J \\
&= \textit{Largest Fix Point of } \Omega.
\end{aligned}
$$

Define

$$
\begin{aligned}
L_0 &= L \\
L_{j+1} &= \Omega(L_j) \\
L_0 &\supseteq L_1 \supseteq L_2 \supseteq \cdots \supseteq L_j \supseteq L_{j+1} \supseteq \cdots \ \textit{(Monotone)} \\
L^{\uparrow} &= \bigcap_{j=0}^{\infty} L_j = \lim_{j \to \infty} L_j = \textit{Largest fixed point of } \Omega.
\end{aligned}
$$

Let

$$
\begin{aligned}
\|L\| &= \textit{Minimal \# of states of an FSA recognizing } L. \\
\|M\| &= \textit{Minimal \# of states of an FSA recognizing } M.
\end{aligned}
$$

Then

$$\forall_i \qquad \|L_i\| \leq \|L\| \cdot \|M\| + 1$$

and $\forall_{j \geq B}(\|L\|, \|M\|)$

$$L_j = L_{j+1} = L_{j+2} = \cdots = L^{\uparrow}$$

(That is descending chain of languages is stationary)

Note, it can be shown that

$$B(\|L\|, \|M\|) = 0(\|L\| * \|M\|)$$

$\therefore$

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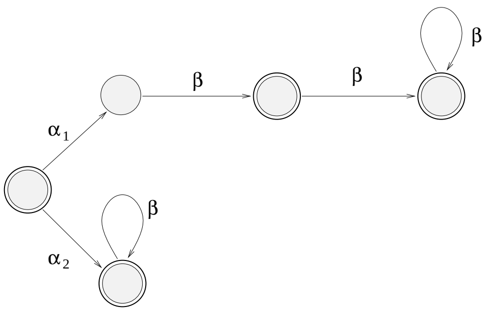

Theorem 7.1.2 *Given a plant G with behavior (language) $M = L(G)$ and a sublanguage $L \subseteq M$, in polynomial time (in $\|L\|$ and $\|M\|$), one can construct a supervisor (S, η) s.t.*

$$L_\eta(G) = L^\uparrow$$

= the supremal controllable sublanguage of L.

Example 7.1.2

$$\begin{array}{rcl} M &=& \overline{(\alpha_1\beta^2 + \alpha_2)\beta^*} \\ L &=& \overline{\alpha_1\beta^2} + \alpha_2\beta^* \end{array}$$

$$\begin{array}{lcl} L_0 &=& L = \overline{\alpha_1\beta^2} + \alpha_2\beta^* \\ L_1 &=& \Omega(L_0) = \{t\colon\ t \in L \ \mathit{and}\ \bar{t}\beta \cap M \subset \bar{L}_0\} \\ &=& \overline{\alpha_1\beta} + \alpha_2\beta^* \\ L_2 &=& \Omega(_1) = \{t\colon\ t \in L \ \mathit{and}\ \bar{t}\beta \cap M \subset \bar{L}_1\} \\ &=& \bar{\alpha}_1 + \alpha_2\beta^* \\ L_3 &=& \Omega(L_2) = \epsilon + \alpha_2\beta^* \\ L_4 &=& \Omega(L_3) = \epsilon + \alpha_2\beta^* = L_5 = L_6 = \cdots = L^\uparrow \\ L^\uparrow & = \epsilon + \alpha_2\beta^* & \end{array}$$

7.2 Modular Synthesis

⋄ Modular problem specification and

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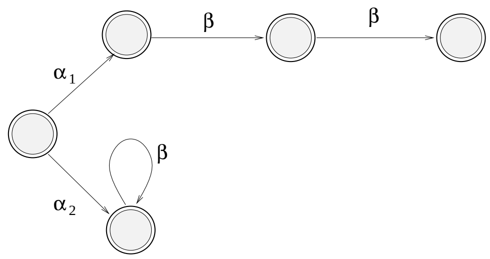

◇ Modular supervisor construction

Example:

◇ It is desired to control a DES so as to specify two constraints simultaneously.

◇ Each constraint is specified as a desired controlled language, K_i, $(i = 1, 2)$

⇒ Overall constraint is specified by

$$K_1 \cap K_2.$$

NOTE: $\overline{K_1 \cap K_2} \subseteq \overline{K}_1 \cap \overline{K}_2$

Definition 7.2.1 *K_1 and K_2 are* **nonconflicting** *if*

$$\overline{K_1 \cap K_2} = \overline{K}_1 \cap K_2$$

Lemma 7.2.1 *K_1, $K_2 \subseteq U^*$, nonconflicting K_1 and K_2 are both closed and controllable. Then $K_1 \cap K_2 =$* **closed and controllable**

Lemma 7.2.2 *K_1, $K_2 \subseteq U^*$ and $K_1^\uparrow$, $K_2^\uparrow$ = nonconflicting. Then $K_1^\uparrow \cap K_2^\uparrow = (K_1 \cap K_2)^\uparrow$.*

7.2.1 Modular Construction of Supervisors

If η and θ are supervisors

$$\begin{aligned} \eta &: L \to \Gamma \\ &: w \mapsto r_\eta \end{aligned}$$

and

$$\theta : L \to \Gamma$$

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$$: \quad w \mapsto r_\theta$$

Then their conjunction is $\eta \wedge \theta$ is the supervisor

$$\begin{aligned} \eta \wedge \theta \quad &: \quad L \longrightarrow \Gamma \\ &: \quad w \longmapsto r_\eta \cap r_\theta. \end{aligned}$$

If η and θ have state realizations

$$(S, \phi) \text{ and } (T, \psi)$$

Then the state realization of $\eta \wedge \theta$ is given by

$$(S \times T, \phi \diamond \psi)$$

$\equiv$ S and T are run in parallel and the feedback map is the intersection of the enabled events of ϕ and ψ.

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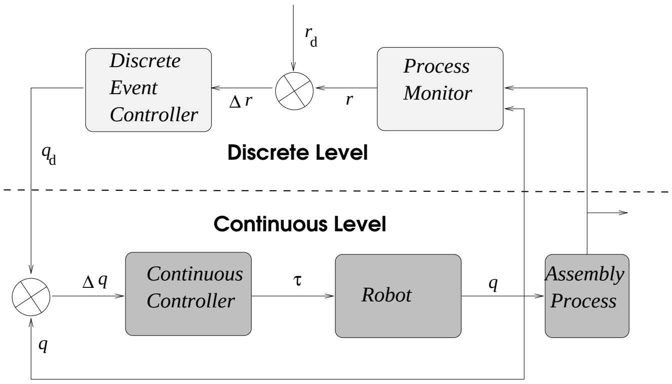
Chapter 8

Robot Assembly

Lecture #8
Mar. 15, 1994[1]

8.1 A Discrete Event Approach to the Control of Robotic Assembly Task.

⋄ Assembly process ≡ Sequence of state changes, where state = set of contacts.
⋄ State changes indicate significant changes in the system dynamics and control.



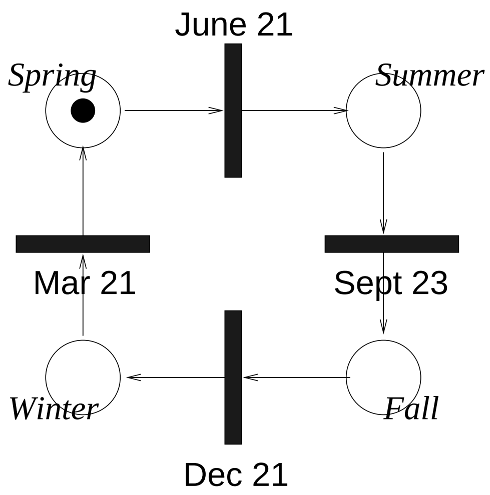
[1] See the article by McCarraghar and Asada (ICRA '93)

8.1.1 Petri Net Model

$$
\begin{aligned}
\mathcal{P} &= \text{Set of Places} \\
\mathcal{T} &= \text{Set of Transitions} \\
\mathcal{I} &= \text{An Input Function, } I\text{: } \mathcal{P} \longrightarrow \mathcal{T} \\
\mathcal{O} &= \text{An Output Function } \mathcal{O}\text{: } \mathcal{T} \longrightarrow \mathcal{P} \\
\#\ \text{Places, } |\mathcal{P}| &= p \\
\#\ \text{Transitions, } |\mathcal{T}| &= c.
\end{aligned}
$$

A petri net can be thought of as a bipartite directed graph $(\mathcal{P} \cap \mathcal{T}, \mathcal{I} \cup \mathcal{O})$, with directed edges $\mathcal{I}$ going from $\mathcal{P}$ to $\mathcal{T}$ and directed edges $\mathcal{O}$ going from $\mathcal{T}$ to $\mathcal{P}$.

Example 8.1.1 4 Seasons Petri Net



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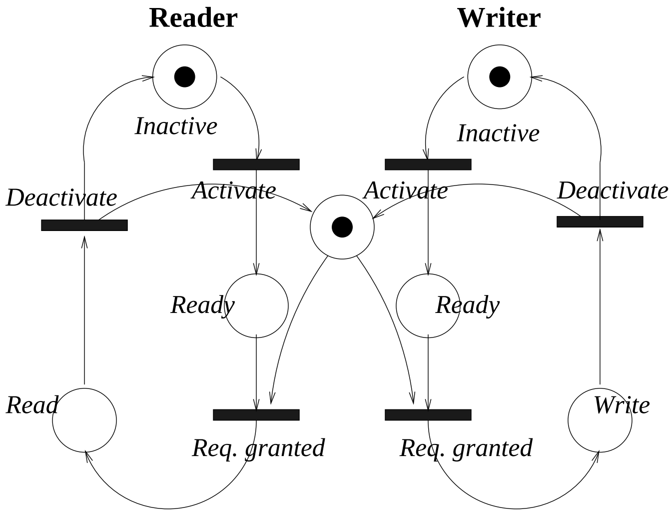
Example 8.1.2 Reader/Writer Petri Net



8.1.2 Controlled Petri Nets

$$\begin{aligned} B &= \text{Set of Discrete control} \\ C &= \text{Control Function, } C\text{: } \mathcal{B} \to \mathcal{T} \text{ (one-to-one)} \end{aligned}$$

\# Discrete controls, $|\mathcal{B}| =$

$$\# \text{ Transitions, } |\mathcal{T} = C.$$

◇ **Places** = Conditions of the current state that enable the transition to occur.
◇ **Controls** = Conditions of the external inputs that enable the transition to occur.
Note: Controls only allow or prevent a transition to occur. They <u>do not force</u> the transition to occur.
$\phi =$ Set of vertices of the workpiece (peg) and fixtures (hole)/constraints.
$\psi =$ Set of edges of the workpiece (peg) and fixtures (hole)/constraints.
Place. $(\phi, \psi) =$ A contact pair

1. An edge of the workpiece $(\epsilon\psi)$ is in contact with a vertex of a fixture $(\epsilon\phi)$.

   <u>OR</u>

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2. A vertex of the workpiece ($\epsilon\phi$) is in contact with an edge of a fixture ($\epsilon\psi$).

   $\therefore$ Each place represents only one constrain equation.

Transition. Each transition represents gain or loss of a single contact.
Note: Combinations of places can be used (e.g. to describe cases of 2-points contact where two constraints are simultaneously active.)
Discrete Control. Discrete controls are defined such that there is one control variable for each transition.

$$\eta \;:\; \mathcal{B} \longrightarrow \{0,1\}$$
$$b \longmapsto \begin{cases} 0 & \text{control is disabled,} \\ & \text{i.e. transition } C(b) \text{ is control disabled.} \\ 1 & \text{control is enabled,} \\ & \text{i.e. transition } C(b) \text{ is control enabled.} \end{cases}$$

Note: The input function defines the places that must be active for a given transition to fine $\Rightarrow$ I.e. the transition is "place enabled". A transition $C(b)$ is "control enabled" iff $\eta(b) = 1$.
Note: A transition that is both "place enabled" and "control enabled" may **fire**.

$$\begin{aligned} r &= \text{Marking} \\ &= \text{Vector assignment of tokens to the places} \\ &= (p \times 1) \text{ column vector (where } p = |\mathcal{P}|) \\ &= \text{Total numbers of possible contact pairs.} \end{aligned}$$

A place is "marked" iff

$$(\phi, \psi) \text{ contact pair for the place is active.}$$

Note: A place may have only 0 or 1 token (i.e. total number of token $\leq p$).

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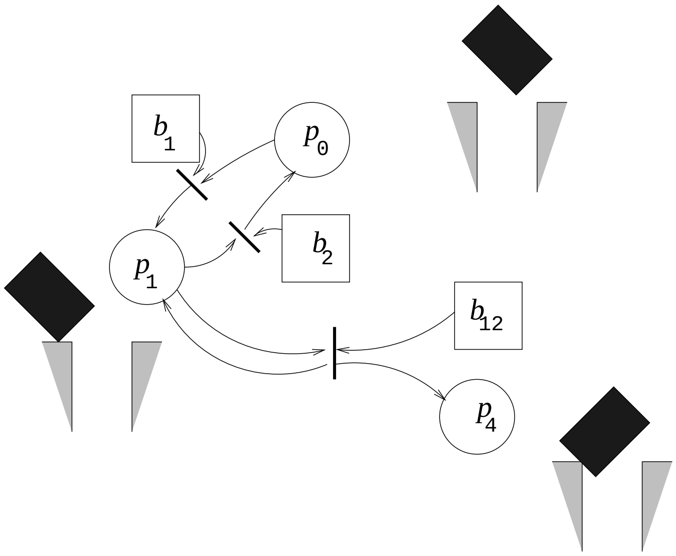
8.1.3 Firing of a Transition

Removes the tokens from each of the (marked) input places and places a token in each of the output places.

⇒ **Next desired marking** $= r_d$.

By enabling various transitions/controls and executing the net, we are able to direct the evolution of the DES.



8.1.4 Petri net model of "Peg-in-the-Hole" problem.

NOTE: Depending on the state (i.e., set of marked places and hence, the set of contacts), we have different constraint equations:

⇒ Different number of DOF (degrees-of-freedom).

⇒ Different number of equations of motion.

◇ "**Maintaining Condition**"

= Condition on the admissible velocity commands that will maintain a current contact.

◇ "**Enabling Condition**"

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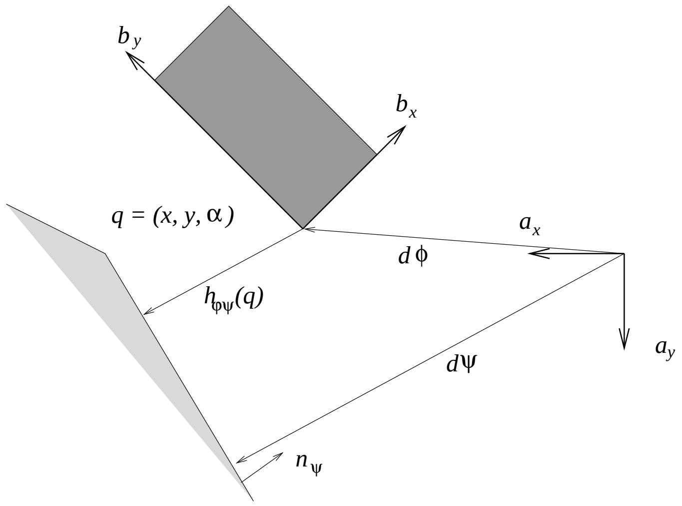
= Necessary condition for a discrete event to occur.

◇ "**Disabling condition**"

= Sufficient condition for a discrete event not to occur



$$
\begin{aligned}
h_{\phi\psi} &= 7(d_\phi - d_\psi) \cdot \eta_\psi \\
&= (d_\phi - d_\psi)^T \eta_\psi
\end{aligned}
$$

contact between the vertex and edge

$$\equiv \qquad h_{\phi\psi} = 0$$

Rate of change of distance (a function of q) is given by:

$$
\begin{aligned}
\frac{dh_{\phi\psi}}{dt} &= \frac{d}{dq}(h_{\phi\psi}) \cdot \frac{dq}{dt} \\
&= \frac{d}{dq}[(d_\phi - d_\psi)^T \eta_\psi] \cdot \dot{q} \\
&= a_{\phi\psi}\dot{q} \equiv a_i \dot{q}
\end{aligned}
$$

where p_i = place corresponding to (ϕ, ψ).

∴ Admissible velocities satisfying the geometric constraints:

$$a_i \dot{q} = 0$$

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$$\begin{aligned} a_i &\triangleq \frac{d}{dq}[(d_\phi - d_\psi)^T \eta_\psi] \\ &= \text{Velocity coefficient vector describing the constraints} \\ &\quad\ \text{represented by } p_i \end{aligned}$$

Maintaining Condition for $\mathbf{p_i}$

$$\equiv \qquad a_i \dot{q} = 0$$

Enabling Condition for $\mathbf{p_i}$

a) **Gain of contact**

$$\equiv \qquad a_i \dot{q} < 0$$

(i.e. $\dot{h}_{\phi\psi} < 0 \ldots$ decreases the distance $h_{\phi\psi}$)
$\Rightarrow$ Places a token in $p_i, \Delta r_i = +1$.

$$(\Delta r_i) a_i \dot{q} < 0$$

b) **Loss of contact**

$$\equiv \qquad a_i \dot{q} > 0$$

(i.e. $\dot{h}_{\phi\psi} > 0 \ldots$ increases the distance $h_{\phi\psi}$)
$\Rightarrow$ Removes a token from $p_i, \Delta r_i = -1$.

$$(\Delta r_i) a_i \dot{q} > 0$$

$\therefore$ Causes the transition specified by Δr_i to fire.

Disabling Condition for $\mathbf{p_i}$

$$\neg[(\Delta r_i) a_i \dot{q} < 0]$$

$$\equiv \quad (\Delta r_i) a_i \dot{q} \geq 0$$

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8.2 Continuous Velocity Map

$\diamond$ Assume that a discrete event trajectory (i.e. desired trajectory) is given:

We wish to determine the continuous velocities in order to achieve the desired final state.

$\diamond$ Suppose we wish to fire a transition

$\Rightarrow$ We know all the enabling conditions for the transition. All other places (neither input nor output to the transition are maintained.

$\diamond$ This process yields a set of equalities and inequalities.

$$(Eqn*) \quad \left\{ \begin{array}{l} \left. \begin{array}{rcl} a_{i_1} \cdot \dot{a} & = & 0 \\ & \vdots & \\ a_{i_1} \cdot \dot{q} & = & 0 \end{array} \right\} \quad \text{Maintaining condn} \\ \\ \left. (\Delta r_{j_1}) a_{j_1} \cdot \dot{q} \;=\; 0 \right\} \quad \text{Enabling condn} \\ \\ \left. \begin{array}{rcl} (\Delta r_{k_1} a_{k_1} \cdot \dot{q} & \geq & 0 \\ & \vdots & \\ (\Delta r_{k_m}) a_{k_m} \cdot \dot{q} & \geq & 0 \end{array} \right\} \quad \text{Disabling condn} \end{array} \right.$$

Max-Min Strategy (Best possible worst case performance)

$$J = \max_{\dot{q}} \left[\min_{i \in \mathcal{A}} a_i \dot{q} \right]$$

subject to (Eqn $*$),

where $\mathcal{A} = \{i_1, \ldots, i_i, j_1, k_1, \ldots, k_m\}$

$\dot{q}_1$ $\dot{q}_3$

$h_{\phi_1 \psi_1}$ $h_{\phi_2 \psi_2}$

q_2

$$\dot{q}_1 \quad : \quad \frac{dh_{\phi_2 \psi_2}}{dt} \ll \frac{dh_{\phi_1 \psi_1}}{dt}$$

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$$\begin{aligned}
\dot{q}_2 &: \quad \frac{dh_{\phi_2\psi_2}}{dt} \approx \frac{dh_{\phi_1\psi_1}}{dt} \\
\dot{q}_3 &: \quad \frac{dh_{\phi_2\psi_2}}{dt} \gg \frac{dh_{\phi_1\psi_1}}{dt}
\end{aligned}$$

$J = \max_{\dot{q}} \left[\min_{i \in \mathcal{A}} a_i \dot{q} \right]$ is maximized/optimized for the value $\dot{q}_2$: A compromise that tries to get both contacts roughly simultaneously.

NOTE: The velocity may be unbounded in some direction. We get around this problem by adding another "velocity constraint".

$$\|\dot{q}\|^2 = (\dot{x})^2 + (\dot{y})^2 + (\ell\dot{\alpha})^2 \leq \text{const.}$$

Note: If there is no feasible velocity solution, then the workpiece has no degrees of freedom and it is "**wedged**".

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