Due by June 30. Collaboration is allowed; please mention your collaborators. Before embarking on any proof, first write what it is that you want to prove in a formal way using what you learned from logic. In addition to that, always mention what kind of statement it is (universal statement, existential conditional statement, . . . ) and what kind of proof you used.

Problem 1
Use direct proof to show that the sum of an odd integer and an even integer is odd.

Problem 2
Prove that if \( m \) and \( n \) are integers and \( mn \) is even, then \( m \) is even or \( n \) is even.

Problem 3
Prove that if \( n \) is a positive integer, then \( n \) is even iff (if and only if) \( 7n + 4 \) is even.

Problem 4
Prove that for all integers \( m \), \( 7m + 4 \) is not divisible by 7.

Problem 5
Prove that there is no \( i \) greatest even integer, \( ii \) least positive rational number.

Problem 6
Prove that if the sum of two real numbers is less than 80, then at least one of the numbers is less than 40.

Problem 7
The following “proof” that every integer is rational is incorrect. Find the mistake.

“Suppose not. Suppose every integer is irrational. Then the integer 1 is irrational. But 1=1/1, which means it is rational. This is a contradiction. Hence the supposition is false and the theorem is true.”

Problem 8
Prove that 2 divides \( n^2 + n \) whenever \( n \) is a positive integer.

Problem 9
Prove that \( 1^2 + 3^2 + 5^2 + \cdots + (2n + 1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3} \).

Problem 10
Prove that a set with \( n \) elements has \( \frac{n(n-1)(n-2)}{6} \) subsets containing exactly three elements whenever \( n \) is an integer greater than or equal to 3.
Problem 11
Determine whether each of these sets is the power set of a set, where $a$ and $b$ are distinct elements.

a) $\emptyset$

b) $\{\emptyset, \{a\}\}$

c) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$

d) $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

Problem 12
Find two sets $A$ and $B$ such that $A \in B$ and $A \subseteq B$.

Problem 13
Let $A, B$ and $C$ be sets. Show that i) $(A \cup B) \subseteq (A \cup B \cup C)$ and ii) $(A \cap B \cap C) \subseteq (A \cap B)$.

Problem 14
Give an example of a function from $\mathbb{N}$ to $\mathbb{N}$ that is

a) one-to-one but not onto.

b) onto but not one-to-one.

c) both onto and one-to-one (but different from the identity function).

d) neither one-to-one nor onto.

(Hint: You don’t need to write the function as a formula)