Prime Numbers in Music Tonality

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1 The Need for Tonality

Music can be defined most generally as organized sound. This definition can be abstracted to the organization of time into patterns as perceived by the sensors in the ear. Sound as we understand it is comprised of two components: changes in air pressure that are easily measured and studied, and the phenomenon of what happens in the brain when these fluctuations in air pressure are converted to electrical signals and digested by our consciousness. The science is solid and comprehensive on the former but only very elemental regarding the latter. This paper will combine our knowledge of the measurable side of music with some principles of basic math to better understand why patterns in music have evolved the way they have and why, especially in western music, out of the infinite ways to organize sound, one system of organization has prevailed, mostly unchallenged.

Human hearing is generally crediting with capturing the frequency range of 20 to 20,000 Hertz. For most, the range is more like 50 to 15,000 Hertz. An absence of energy is referred to as silence and all frequencies with equal energy heard simultaneously is called white noise. As far as music is concerned, everything in between is free range for musical composition. Since no practical sound is made of single frequency (even a single sine wave resonates in the ear canal creating overtones), the basic component of music- a single tone- is based upon some relationship of frequencies. Furthermore, since most music consists of more than one single tone, what set of tones should be used? Either as a monophonic melody or a composite harmony of sound, music is based on the ratios of frequencies, and the composer must choose these ratios. This is known as tonality: a framework of ratios upon which to arrange sound. As Haluska puts it: "The sculptor working in marble has set his limit by the choice of this material to the exclusion of all other materials. Analogously, the musician has to select the tone system he wants to use." [Haluska, iii]

2 The Common Systems

Given the seemingly unlimited number of tonalities that can be built, it would be useful if there was a way to classify these different systems. One way to do this is to look at any given tonality in terms of its prime limit: any tonality is comprised as a set of ratios, and by the Fundamental Theory of Arithmetic, these ratios can be expressed as a product of prime numbers. The largest prime number found in any set of ratios is called the Limit of that system.

2.1 Pythagoras and the Monochord: Unity and Equivalency

2.1.1 Prime Unity

The monochord is an early musical instrument that is attributed to Pythagoras. Like a modern string instrument, it consists of a string that is pulled tight at both ends and suspended over a resonating body. When plucked, the instrument makes a sound with a pitch proportional to the length, width and tension of the string.

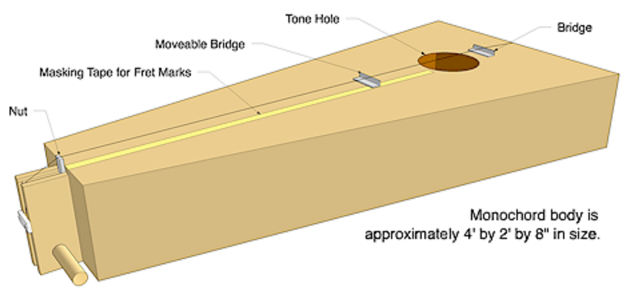

Figure 1: The Monochord [Building a Monochord]

The pitch created by plucking the open string is the lowest pitch that can be generated by the instrument and is therefore the basis of expansion for every ratio in the system. This is referred to as the unison, or Prime Unity, and is denoted as the fraction $\frac{1}{1}$ [Partch, 71]. Two monochords with the same specifications will make the same lowest pitch, and when played in unison they blend together in the ear such that they are indistinguishable.

2.1.2 The Octave, or 2-Limit

To generate other pitches, the monochord has a moveable bridge that slides from one end of the string to the other. Pythagoras noticed that if he placed the bridge in the middle of the string, creating two sections of string with equal width and tension but half the length, they had an interesting relationship to the fundamental pitch of the instrument. To him and others, this could only be explained as a consonant blending, an effect that they were the same sound in different places. "It is one of the amazing phenomena of acoustics that the 2/1 of a tone, the doubling of its cycles, gives a tone which we know instantly to be different from the first, but one so like the first that we deny it a separate identity." [Partch, 88] This gave birth to what is now known as octave equivalency and is the first step of the N-Limit concept: any pitches that are related by a factor of two are considered to have the same musical value, and are therefore the same note. Today we know that because of the way a string vibrates, its harmonic motion means that it also vibrates in a series of waves that are whole number ratios of the fundamental and that the second mode, or first harmonic, with a ratio of 2:1 is the strongest of these modes. Not surprisingly, the next mode, or 3 times the fundamental is the next strongest of these modes. In Figure 2 we see the harmonic ratios of the fundamental represented with amplitude values. If we take that series and move it up an octave, each harmonic of the upper octave lines up with a harmonic of the lower octave. This may be the fundamental cause of this equivalency effect.

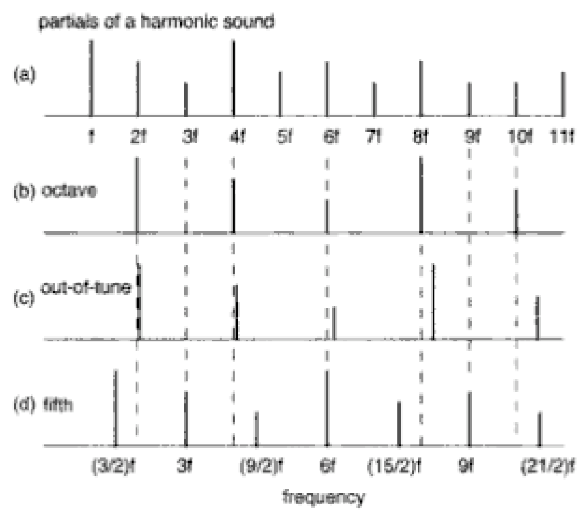

Figure 2: Octave equivalency in terms of harmonics [Sethares, 53]

2.1.3 Making Cents

Since octave equivalence is based on the doubling of a frequency, it makes more sense to measure intervals with a log-based system. In 1830 Alexander Ellis proposed a system where intervals could be added and subtracted by taking the base-2 log of their ratio and multiplying that by a set amount of octave divisions. Since there were 12 steps to the octave, he made the octave worth 1200 cents, with 100 cents for each scale step. This is a useful system for comparing the small differences in ratios on a linear scale. [Sethares, 331] His formula can be represented as:

$$c = 1200 \log_2(\frac{a}{b}) \tag{1}$$

or:

$$c = \Big(\frac{1200}{\log_{10}(2)}\Big) * \log\Big(\frac{a}{b}\Big) \tag{2}$$

Therefore, the number of cents between 200 Hertz and 300 Hertz is simply:

$$c = 1200 \log(\frac{300}{200}) = 1200 * 0.5849625 = 701.955$$

To convert from cents to the ratio it represents:

$$r = 10\Big(\frac{c \log_{10}(2)}{1200}\Big) \tag{3}$$

or approximately:

$$10^{0.00025086c} \tag{4}$$

1200 cents in the octave provides enough resolution to describe intervals at the limits of human perception (human ears cannot distinguish intervals less than approximately 30 Hertz). From here on we will use this formula to create another reference for comparing ratios.

2.2 The Pythagorean Cycle, or 3-Limit

Given the consonance inherent in the 2:1 ratio, Pythagoras believed that the next most consonant, or pleasing, interval should be 3:1. In discussing ratios it is easiest to compare them if they are in the same number range which is between the unison and octave, or between 1 and 2. Because of octave equivalency any pitch that is greater than two times the unison is just a repetition of a pitch between 2:1 and 1:1 and it is easiest to divide this ratio by two until it is in the appropriate range. In prime factor notation, this can be expressed as 2^{-1}:

$$\frac{2}{1} = 2^1$$

$$\frac{3}{1} = 3^1; \frac{3}{2} = \frac{1}{2} * \frac{3}{1} = 2^{-1} * 3^1$$

Pythagoras believed that he could construct the best tone system based on the ratio 3:2. Because of octave equivalency, the ratio 3:2 is also equivalent to the ratio of 3:4. Alternatively, if we create a point 3:2 of the way between octaves, another 4:3 will bring us to the next octave:

$$1 * \frac{3}{2} * \frac{4}{3} = 2$$

and therefore one interval does not exist without the other. If we take the unison and multiply it by successive ratios of 3:2 and 4:3, we end up with what is called the Pythagorean Cycle:

Unison	$*\frac{4}{3}$	$*\frac{4}{3}$	$*\frac{4}{3}$	$*\frac{4}{3}$	$*\frac{4}{3}$	
1:1	4:3	16:9	32:27	128:81	256:243	
Unison	$*\frac{3}{2}$	$*\frac{3}{2}*2^{-1}$	$*\frac{3}{2}$	$*\frac{3}{2}*2^{-1}$	$*\frac{3}{2}$	$*\frac{3}{2}*2^{-1}$
1:1	3:2	9:8	27:16	81:64	243:128	729:512

Table 1: Pythagorean Cycle

Today this cycle is still the basis of western musical theory and is known as the cycle of fifths (because the ratio 3:2 has become the fifth step in the scale):

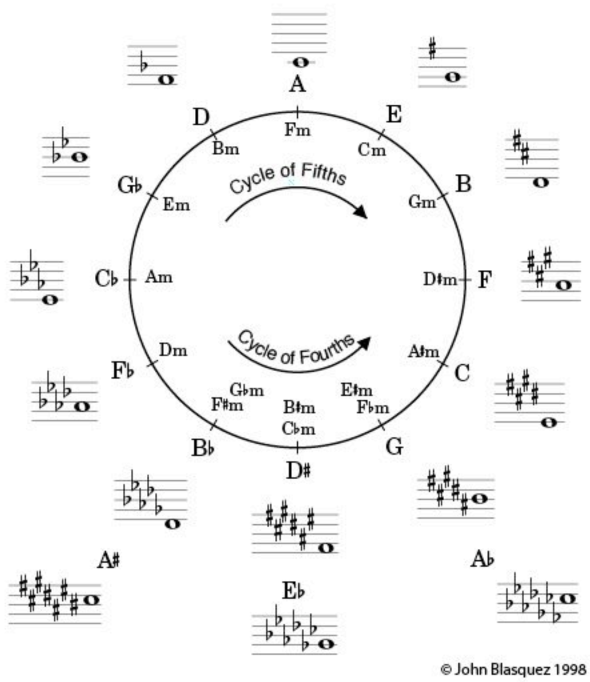

Figure 3: The Cycle of Fifths [Blasquez]

If we took the seventh step of 3:2, we would get

$$\frac{729}{512} * \frac{3}{2} = \frac{2187}{2048} \approx 1.068$$

which is almost equivalent to the unison, and is where the cycle repeats itself. Another way to see this is:

$$(\frac{3}{2})^{12} = \frac{531441}{4096} * 2^{-7} = \frac{531441}{524288} \approx 1.0136$$

The small difference between these octaves is called the Pythagorean comma, or wolf tone. Although this inconsistency is a flaw in the system, it is possible to diminish one of the values of 3:2 by this amount and avoid that interval in a musical piece, or distribute the difference among all the ratios of 3:2 [Sethares, 52-5].

If we arrange these scale steps in ascending order, we get the following:

Ratio	Prime Notation	Decimal	Cents
1:1	2^0	1	0
256:243	$2^8 * 3^{-5}$	1.0535	90.22
9:8	$2^{-3} * 3^2$	1.125	203.91
32:27	$2^5 * 3^{-3}$	1.18512	294.13
81:64	$2^{-6} * 3^4$	1.2656	407.82
4:3	$2^2 * 3^{-1}$	1.3333	498.04
729:512	$2^{-9} * 3^6$	1.4238	611.73
3:2	$2^{-1} * 3^1$	1.5	701.96
128:81	$2^7 * 3^{-4}$	1.5802	792.18
27:16	$2^{-4} * 3^3$	1.6875	905.87
16:9	$2^4 * 3^{-2}$	1.77778	996.09
243:128	$2^{-7} * 3^5$	1.8984	1109.78

Table 2: Pythagorean Intervals Ascending

If we select certain ratios from this set, we get the following scale:

Ratio	$\frac{1}{1}$		$\frac{9}{8}$		$\frac{81}{64}$		$\frac{4}{3}$		$\frac{3}{2}$		$\frac{27}{16}$		$\frac{243}{128}$		$\frac{2}{1}$
Step		$\frac{9}{8}$		$\frac{9}{8}$		$\frac{256}{243}$		$\frac{9}{8}$		$\frac{9}{8}$		$\frac{9}{8}$		$\frac{256}{243}$	

Table 3: Pythagorean Scale

The strength of this scale lies in the fact that there are two consistent values between scale steps: 9:8 and 256:243. The consistency between "half" and "whole" steps, as they will later be called, provides a symmetry that is important in creating melodies. Pythagoras was convinced that music composed with small simple ratios was superior and although this scale is based on the simple ratio of 3:2, the cycle builds some very complex ratios that are not the most appealing to the ear. Today there are still 12 steps in the tone system and 8 in the natural scale, but theorists found that by adding another prime to the system more simple, and therefore more pleasing scale steps could be formed.

2.3 The 5-Limit

By adding 5 to the system we get the following commonly used ratios: 5:4, 6:5, 5:3, 8:5, 9:5 and 15:8. The 3-Limit system can now be expanded to:

Ratio	Prime-Factor Notation	Decimal	Cents
1:1	1	1	0
16:15	$2^4 * 3^{-1} * 5^{-1}$	1.067	111.73
10:9	$2^1 * 5^1 * 3^{-2}$	1.111	182.40
9:8	$3^2 * 2^{-3}$	1.125	203.91
32:27	$2^5 * 3^{-3}$	1.185	294.13
6:5	$2^1 * 3^1 * 5^{-1}$	1.2	315.64
5:4	$5^1 * 2^{-2}$	1.25	386.31
4:3	$2^2 * 3^{-1}$	1.333	498.04
45:32	$3^2 * 5^1 * 2^{-5}$	1.406	590.22
3:2	$3^1 * 2^{-1}$	1.5	701.96
8:5	$2^3 * 5^{-1}$	1.6	813.69
5:3	$5^1 * 3^{-1}$	1.667	884.36
16:9	$2^4 * 3^{-2}$	1.778	996.09
9:5	$3^2 * 5^{-1}$	1.8	1017.60
15:8	$3^1 * 5^1 * 2^{-3}$	1.875	1088.27
2:1	2^1	2	1200

Table 4: Common 5-Limit Intervals

These ratios of 5 were recognized as scale degrees as early as the fourth century BC [Partch, 91]. Paul Erlich has another way of showing the 5-Limit, in terms of the groups of major and minor triads:

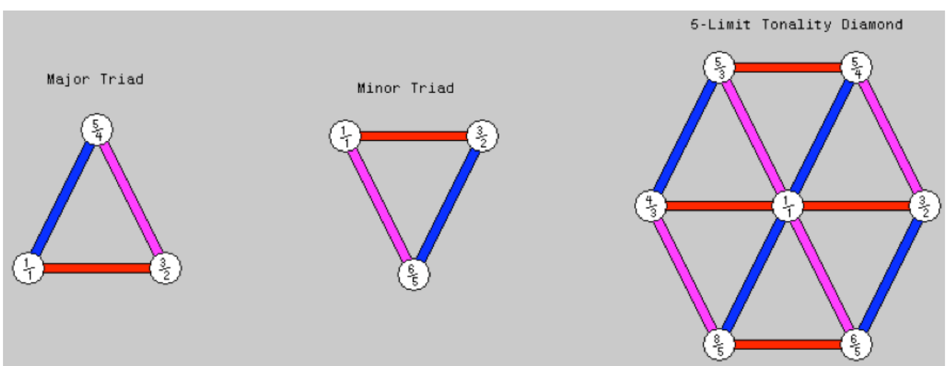

Figure 4: 5-Limit Tonality Diamond [Erlich, 6]

2.4 Just Intonation vs. Equal Temperament

All of the ratios that have been accepted as the basis for tonality in Western Music for the last several hundred years can be found within the 5-Limit system. However, although these ratios serve as the basis you will not actually find one in the current system. Tonality based on whole numbers has become known as Just Intonation, and although whole number interval ratios may be the most appealing to the ear with harmonic instruments, the system has flaws that render it unacceptable for creating melodies. As seen in Table 4, no two intervals have the same spacing between them. This is fine if the music has only one pitch center (known as the key), but music rarely does. Imagine that you create a melody with the first few scale steps: 1:1, 9:8 and 6:5. Now if you want to use that same melody starting on a different step in the scale, such as 3:2, you have two undesirable choices. If you use the exact same spacing between the notes, you are creating pitches that were not in the scale to begin with:

$$\frac{3}{2} * \frac{9}{8} = \frac{27}{16} = 1.6875$$

This is close to, but not exactly 5:3 (1.667). If you use 5:3 instead of 27:16 as the step you remain in the key, but the interval between your notes has changed and the melody will not sound the same. The problem is compounded when scale steps are played simultaneously as chords and as composers became more adventurous with dense chords and key changes in their work, this became unacceptable. Composers experimented with ways to adjust the system to minimize these problems, such as Well Temperament and Meantone Temperament, but it became clear that the only way to solve this problem is to choose a smallest interval in the scale and make every other interval an even multiple of it.

It is unclear when this was first proposed, but it can be traced in western culture back to the second half of the 16th century [Wikipedia]. In Equal Temperament, each scale step must be the same ratio between 1 and 2. As the octave had already been divided into 12 steps, each interval becomes the same ratio: $\sqrt[12]{2} \approx 1.059$. If we plug this into Formula 1 on page 3, we see that each step is exactly 100 cents:

$$1200 \log_2 1.059 = 100$$

The following table compares the intervals of the 5-Limit system with their current approximation in Equal Temperament:

Ratio	Dec.	Cents	ET Ratio	ET Dec.	Cents	Δ Cents
1:1	1	0	1	1	0	0
256:243	1.0535	90.22	$(\sqrt[12]{2})^1$	1.05946	100	+9.78
9:8	1.125	203.91	$(\sqrt[12]{2})^2$	1.12246	200	-3.91
6:5	1.2	315.64	$(\sqrt[12]{2})^3$	1.1892	300	-15.64
5:4	1.25	386.31	$(\sqrt[12]{2})^4$	1.25992	400	+13.69
4:3	1.333	498.04	$(\sqrt[12]{2})^5$	1.33484	500	+1.96
729:512	1.424	611.73	$(\sqrt[12]{2})^6$	1.41421	600	-11.73
3:2	1.5	701.96	$(\sqrt[12]{2})^7$	1.4983	700	-1.96
8:5	1.6	813.68	$(\sqrt[12]{2})^8$	1.5874	800	-13.68
5:3	1.6667	884.36	$(\sqrt[12]{2})^9$	1.6818	900	+15.64
16:9	1.778	996.09	$(\sqrt[12]{2})^{10}$	1.7818	1000	+3.91
243:128	1.898	1109.78	$(\sqrt[12]{2})^{11}$	1.88775	1100	-9.78
2	2	1200	$(\sqrt[12]{2})^{12}$	2	1200	0

Table 5: 5-Limit Just Intonation vs. 12-Tone Equal Temperament

The 5-Limit system has what we now consider to be "pure" intervals, but it is also not approximated as well with 12-step equal temperament as the Pythagorean scale.

3 7, 11 and Beyond

3.1 The 7-Limit

Again Erlich shows us a way to arrange 7-Limit intervals in term of groups, in this case tetrads:

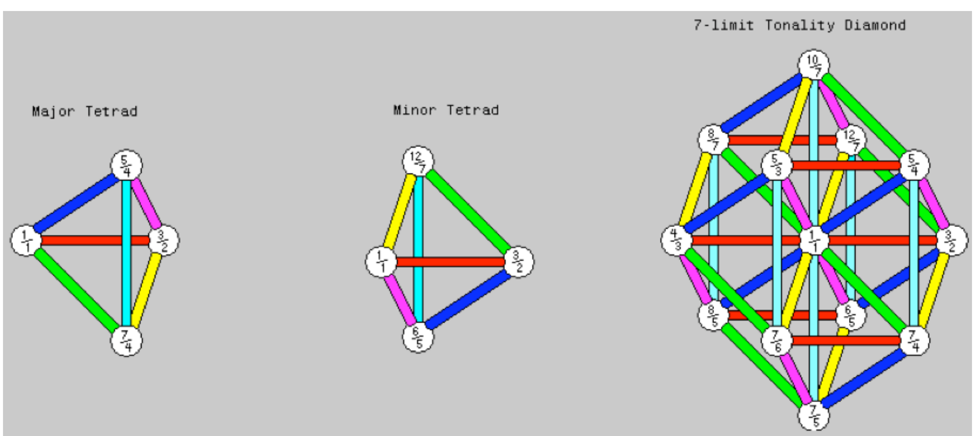

Figure 5: 7-Limit Tonality Diamond [Erlich, 6]

3.2 11-Limit

Harry Partch skipped over the 7-Limit to work with the 11-Limit system. He constructed a 43-tone scale based on this limit which he called the One Footed Bride:

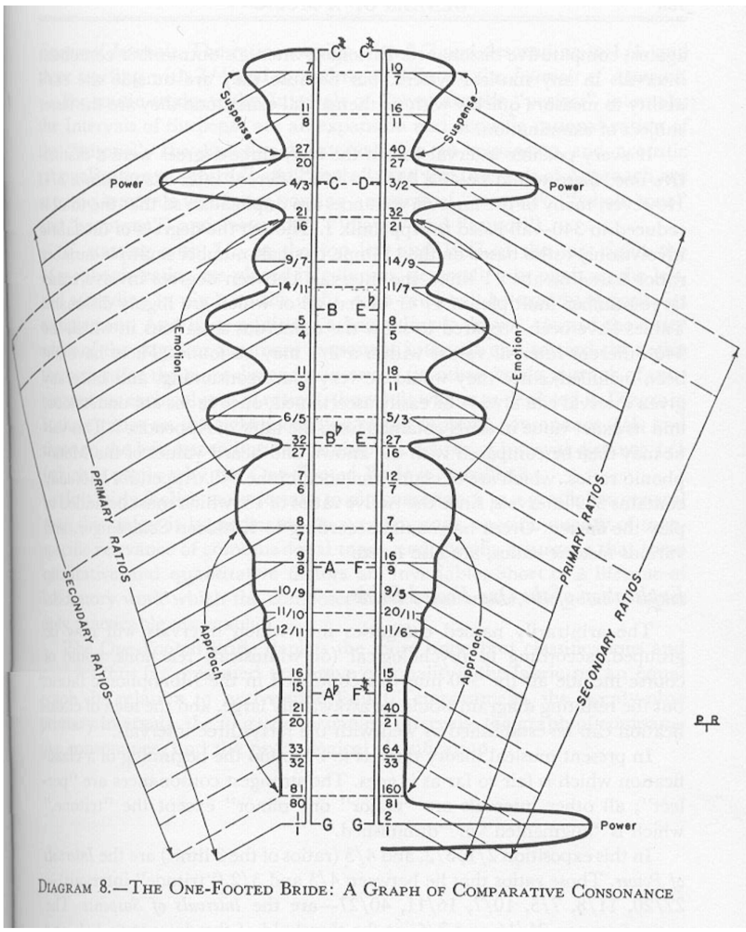

Figure 6: Partch's One Footed Bride [Partch, 155]

Next to the scale steps he has drawn his interpretation of the dissonance of each interval.

3.3 Beyond

It should be clear that any prime number can be used as a basis for a musical tone system. Even though most of western music has settled with the 5-Limit system, many composers have expanded the possibilities of tonality in their compositions by using higher limits. In other cultures, such as those of India and Indonesia, larger N-Limit systems have been the standard all along.

However, the real issue lies in the debate between Just Intonation and Equal Temperament. Many people will argue that Just Intonation is not practical for any musical purposes and is therefore unusable. Others, in the Pythagorean spirit, say that Equal Temperament is lacking because it is not based on “pure” simple ratios. In Equal Temperament the question is not what prime limit to use, but how many evenly spaced steps to use. One can certainly build a scale out of whole ratios and then equally temper the results, but this may defeat the purpose of the whole ratios in the first place.

The concept of dissonance has played the fundamental role in shaping the sets of ratios in our musical systems. But given what we know about musical acoustics and what we can do with computers, we can use this framework in a whole new context. Whereas before mathematicians and composers chose scale ratios based on the harmonic qualities of early instruments, we can now reverse this process by creating a tone system with a set of ratios and then engineering an instrument to match these qualities. This is particularly well suited for computer synthesis because sounds can be built by directly adding together sine waves and “instruments” can be optimized to work within a given set of ratios. Others have created algorithms to adapt the intervals of Just Intonation in real time to compensate for the lack of symmetry.

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