The Parallel Postulate

For more than 2,000 years, the five postulates of Euclid’s *Elements* were accepted as truths. They are, in modern terms:

I. A straight line may be drawn between any two points.
II. A line segment may be extended indefinitely.
III. A circle may be drawn with any given radius and an arbitrary center.
IV. All right angles are equal in measure.
V. Parallel Postulate: Given a line L and a point P not on line L, there exists exactly one line in the plane of point P and line L that contains point P and is parallel to line L (Henle 8).

Mathematicians believed, up to the 1800s, that Euclid’s fifth postulate could be proved using the other postulates. If such a feat could be accomplished, the fifth postulate would become a theorem. One of the more notable attempts to show that the parallel postulate is a logical consequence of the other postulates was made in 1733 by Italian Jesuit priest and mathematician Girolamo Saccheri. Saccheri spent a lifetime preparing his proof *Euclid Freed of All Blemish*, but even a lifetime of work could not prevent its inevitable failure.

Although Saccheri’s proof was flawed, the real prize of his work was the Saccheri Quadrilateral, which motivated the rise of two different branches of non-Euclidean geometry—hyperbolic and elliptic geometry—that are just as consistent as Euclidean geometry (MacDonnell).

On that note, let us begin by examining the properties of a Saccheri Quadrilateral.

A Saccheri Quadrilateral is a quadrilateral ABCD where $BAD$ and $ABC$ are right
angles and $\overline{AD} \equiv \overline{BC}$. Segment $\overline{AB}$ is called the base and segment $\overline{CD}$ is called the summit (Henle 10).

It’s clear that the existence of the Saccheri Quadrilateral depends only on the first four of Euclid’s postulates, but what’s less apparent is Saccheri’s intention. That is to say, Saccheri was trying to construct the Parallel Postulate from the first four geometrically. If he could prove that the summit angles were either both right, or supplementary, then he would have proven the Parallel Postulate a theorem.

The problem, however, is the nature of the summit angles. By theorem, a quadrilateral with at least one right angle and one pair of congruent sides has congruent diagonals. So we construct diagonals $\overline{AC}$ and $\overline{BD}$:

From the definition, $\overline{AD} \equiv \overline{BC}$; we concluded that $\overline{AC} \equiv \overline{BD}$; and $\overline{CD} \equiv \overline{CD}$ by the reflexive property of congruence. By Side-Angle-Side congruence, $\triangle ADC \equiv \triangle BCD$.

Therefore, $\triangle ADC \equiv \triangle BCD$, or the summit angles are always congruent.
Such a result might seem insignificant at first, but a closer look at its consequences tells us otherwise. If one of the summit angles is right, then the other is right as well, which implies that the resulting figure is the Euclidean rectangle that we all know and love. On the other hand, taking the summit angles to be acute produces a quadrilateral with angle sum less than $360^\circ$, and, in turn, triangles with angle sum less than $180^\circ$. Similarly, taking the summit angles to be obtuse produces a quadrilateral with angle sum greater than $360^\circ$.

Both the acute and obtuse cases seem to be blatant violations of the Parallel Postulate and all that we understand about Euclidean geometry, and violations they are indeed. In fact, these violations provided the incentive for the development of non-Euclidean geometries.

But it wasn’t until the 1800s when Hungarian mathematician Janos Bolyai and Russian mathematician Nikolai Lobachevsky revisited Saccheri’s Quadrilateral and joined forces to develop hyperbolic geometry—a geometric system that conforms to Euclid’s first four postulates, but rejects the Parallel Postulate (Smith 127). Shortly after, mathematician Bernhard Reimann developed elliptic geometry, which also rejects the Parallel Postulate. While Bolyai and Lobachevsky were motivated by the idea of parallelism, Reimann
focused on the metric properties of elliptic geometry; as such, it is to mathematician William Clifford that “we owe the theory of parallels in elliptic space” (Miller 376).

The non-Euclidean geometry contains nothing in it that is contradictory, although at first view very many of the results have the air of paradoxes. These apparent contradictions must be regarded as the effect of an illusion, due to the habit we have of considering the Euclidean geometry as rigorous. —Carl Friedrich Gauss, 1846

Gauss certainly sets the mood as we turn to parallelism in these alternate geometries. Let’s begin by examining the premise of hyperbolic geometry—that is, the notion that a straight line maintains its parallelism at all points (Miller 376).

Let $AB$ be parallel to $CD$ at $E$, and let $F$ be an arbitrary point on $AB$ on either side of $E$. Since $A$, $B$, $E$, and $F$ are collinear, $AB$ is also parallel to $CD$ at $F$.

Now draw segments $EH$ and $FH$ to a point $H$ on $CD$.

If we move $H$ along $CD$, $EH$ and $FH$ approach positions of parallelism with $CD$ as $H$ moves infinitely further away from points $C$ or $D$. By transitivity, we have that $AB$, $CD$,  

---

1 (qtd. in Miller 375)
\(\overline{EH}\), and \(\overline{FH}\) are parallel to each other. Thus, in hyperbolic geometry, there exists more than one line that is parallel to another\(^2\).

In fact, there are two types of parallelism in hyperbolic geometry. But first let’s define a new plane that is along the lines of the Klein model, the simplest model of hyperbolic geometry—that is, the unit disk, or the interior of the unit circle.

Here, hyperbolic straight lines are circles, or sections of circles that intersect the unit disk at right angles. Points on the boundary of the circle are called ideal points; hyperbolic lines are called parallel if they do not intersect inside the unit disk but do share an ideal point; and hyperbolic lines are called hyperparallel if they do not intersect inside the unit disk and do not share an ideal point (Henle 82).

Strangely enough, in hyperbolic geometry, two lines that are hyperparallel to a given line can intersect. The same applies to parallel lines that share an ideal point (the intersection with the circle at infinity). After years of being subjected to the laws of Euclidean geometry, the idea that there exists an infinite number of lines parallel with

\(^2\) This proof was adapted from (Miller 376).
respect to each other can be quite disturbing. But that’s only because most of us aren’t used to seeing a geometry with negative curvature!

On the other hand, what happens if we take the boundary of the unit disk and revolve it around a Euclidean diameter?

We get the polar opposite of hyperbolic geometry—that is, a projection of the points in the unit disk onto the unit sphere, or the simplest model for elliptic geometry!

In elliptic geometry, there is no such thing as a straight line since it is impossible to draw a line with no curvature (a Euclidean line) on a positively curved surface. Instead, there are “straightest” lines, which are commonly referred to as great circles. Great circles are circles drawn on the surface of a sphere whose center is also the center of the sphere (Henle 116). If their centers are concurrent, then great circles are also the circles on the sphere with the largest radius. In essence, a Euclidean straight line would be associated with a sphere of infinite radius, since Euclidean lines are infinitely long in opposite directions. The lack of curvature in Euclidean space is precisely the reason why a straight line cannot intersect itself. But because the elliptic geometry of the sphere depends on the projection of points on the boundary of the unit disk, every point on the sphere has an associated diametrically opposite point—much like the North and South Poles on the Earth. Elliptic lines, or great circles, are therefore self-intersecting, or finite! In fact, since great circles are merely cross sections of the unit sphere, their length is equivalent to the circumference of the unit disk, $\pi$. 
Logically, if elliptic lines are finite in length, then it would seem that there would be less of a chance that one or more elliptic lines could intersect. After all, they do not have all of infinity to find a common point! But exactly the opposite is true; elliptic great circles always intersect, which implies that in elliptic geometry there are no parallel lines at all.

More formally, let $\alpha$ and $\beta$ be two great circles on the surface of the unit sphere. If we fix the center of the sphere at the origin in 3-space, and consider the two planes defined by $\alpha$ and $\beta$, we find that $\alpha$ and $\beta$ always intersect in a Euclidean line, or the diameter of the sphere. It’s clear that two of the points on the Euclidean line are on the surface of the sphere and on $\alpha$ and $\beta$. Therefore, two arbitrary lines in elliptic geometry always intersect at a point and that point’s diametrically opposite point; hence the impossibility of parallel lines in elliptic geometry (Henle 122).
Alternately, the image below demonstrates how elliptic geometry rejects the first and fifth of Euclid’s postulates. The left side is of the form we are used to, that is, the sphere; but right side is a stereographic projection of the unit sphere. Basically, a stereographic projection is a conformal mapping, meaning that it maintains angle measurement, of the points on the unit sphere onto the complex plane. While the projection is not necessary for understanding the theory of parallels in elliptic space, it is worth noting because it establishes, at the very least, a different model for elliptic geometry, called the single elliptic model (shown below as an elliptic cycle).
In everyday life, the things we notice the most are the things that are different or variant. Surely anyone would notice if his or her car were red one day and suddenly green the next. On the other hand, the things we are most accustomed to are the things we notice the least; so it comes as no surprise that the realizations of both hyperbolic and elliptic geometry are relatively new. After all, the physical universe in which we live is mostly hyperbolic and elliptic in curvature. Rarely do we find anything that lacks curvature in nature, which is probably why Euclidean, or neutral geometry, was the first to be noticed. But every now and then a great problem like the Parallel Postulate arises, and it inspires humankind to take the most difficult step of all—that is, to challenge the philosophies of the past, and know by way of that challenge that they have added a completely new perspective, a twist on truth, per se, that future generations can now develop and explore.

Works Cited


MacDonnell, Joseph. *Theorems of Girolamo Saccheri and his hyperbolic geometry.*

Fairfield University. 10 July 2007.

<http://www.faculty.fairfield.edu/jmac/sj/sacflaw/sacther.htm>


Smith, Sanderson. *Agnesi to Zeno: Over 100 Vignettes from the History of Math.*

California: Key Curriculum Press, 1996.