Discrete Mathematics
Lecture 1
Logic of Compound Statements

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Logic of Statements

• Logical Form and Logical Equivalence
• Conditional Statements
• Valid and Invalid Arguments
• Digital Logic Circuits
• Number Systems & Circuits for Addition
Logical Form

• Initial terms in logic: sentence, true, false
• Statement (proposition) is a sentence that is true or false but not both
• Compound statement is a statement built out of simple statements using logical operations: negation, conjunction, disjunction
Logical Form

- Truth table
- Precedence of logical operations
- English words to logic:
  - It is not hot but it is sunny
  - It is neither hot nor sunny
- Statement form (propositional form) is an expression made up of statement variables and logical connectives (operators)
- Exclusive OR: XOR
Logical Form

• Truth table for \((\neg p \land q) \lor (q \land \neg r)\)

• Two statements are called logically equivalent if and only if (iff) they have identical truth tables

• Double negation

• Non-equivalence: \(\neg(p \lor q)\) vs \(\neg p \lor \neg q\)

• De Morgan’s Laws:
  – The negation of and AND statement is logically equivalent to the OR statement in which component is negated
  – The negation of an OR statement is logically equivalent to the AND statement in which each component is negated
Logical Form

• Applying De-Morgan’s Laws:
  – Write negation for
    • The bus was late or Tom’s watch was slow
    • -1 < x <= 4

• Tautology is a statement that is always true regardless of the truth values of the individual logical variables

• Contradiction is a statement that is always false regardless of the truth values of the individual logical variables
Logical Equivalence

- Commutative laws: $p \land q = q \land p$, $p \lor q = q \lor p$
- Associative laws: $(p \land q) \land r = p \land (q \land r)$, $(p \lor q) \lor r = p \lor (q \lor r)$
- Distributive laws: $p \land (q \lor r) = (p \land q) \lor (p \land r)$
  $p \lor (q \land r) = (p \lor q) \land (p \lor r)$
- Identity laws: $p \land t = p$, $p \lor c = p$
- Negation laws: $p \lor \lnot p = t$, $p \land \lnot p = c$
- Double negative law: $\lnot (\lnot p) = p$
- Idempotent laws: $p \land p = p$, $p \lor p = p$
- De Morgan’s laws: $\lnot (p \land q) = \lnot p \lor \lnot q$, $\lnot (p \lor q) = \lnot p \land \lnot q$
- Universal bound laws: $p \lor t = t$, $p \land c = c$
- Absorption laws: $p \lor (p \land q) = p$, $p \land (p \lor q) = p$
Conditional Statements

• If something, then something: $p \rightarrow q$, $p$ is called the hypothesis and $q$ is called the conclusion

• The only combination of circumstances in which a conditional sentence is false is when the hypothesis is true and the conclusion is false

• A conditional statements is called vacuously true or true by default when its hypothesis is false

• Among $\land$, $\lor$, $\neg$ and $\rightarrow$ operations, $\rightarrow$ has the lowest priority
Conditional Statements

- Write truth table for: \( p \land q \rightarrow \sim p \)
- Show that \( (p \lor q) \rightarrow r = (p \rightarrow r) \land (q \rightarrow r) \)
- Representation of \( \rightarrow \): \( p \rightarrow q = \sim p \lor q \)
- Re-write using if-else: Either you get in class on time, or you risk missing some material
- Negation of \( \rightarrow \): \( \sim(p \rightarrow q) = p \land \sim q \)
- Write negation for: If it is raining, then I cannot go to the beach
Conditional Statements

• Contraposition of $p \rightarrow q$ is another conditional statement $\neg q \rightarrow \neg p$

• A conditional statement is equivalent to its contrapositive

• The converse of $p \rightarrow q$ is $q \rightarrow p$

• The inverse of $p \rightarrow q$ is $\neg p \rightarrow \neg q$

• Conditional statement and its converse are not equivalent

• Conditional statement and its inverse are not equivalent
Conditional Statements

- The converse and the inverse of a conditional statement are equivalent to each other
- $p$ only if $q$ means $\neg q \Rightarrow \neg p$, or $p \Rightarrow q$
- Biconditional of $p$ and $q$ means “$p$ if and only if $q$” and is denoted as $p \iff q$
- $r$ is a sufficient condition for $s$ means “if $r$ then $s$”
- $r$ is a necessary condition for $s$ means “if not $r$ then not $s$”
Exercises

• Write contrapositive, converse and inverse statements for:
  – If P is a square, then P is a rectangle
  – If today is Thanksgiving, then tomorrow is Friday
  – If c is rational, then the decimal expansion of r is repeating
  – If n is prime, then n is odd or n is 2
  – If x is nonnegative, then x is positive or x is 0
  – If Tom is Ann’s father, then Jim is her uncle and Sue is her aunt
  – If n is divisible by 6, then n is divisible by 2 and n is divisible by 3
Arguments

• An argument is a sequence of statements. All statements except the final one are called premises (or assumptions or hypotheses). The final statement is called the conclusion.

• An argument is considered valid if from the truth of all premises, the conclusion must also be true.

• The conclusion is said to be inferred or deduced from the truth of the premises
Arguments

• Test to determine the validity of the argument:
  – Identify the premises and conclusion of the argument
  – Construct the truth table for all premises and the conclusion
  – Find critical rows in which all the premises are true
  – If the conclusion is true in all critical rows then the argument is valid, otherwise it is invalid

• Example of valid argument form:
  – Premises: \( p \lor (q \lor r) \) and \( \sim r \), conclusion: \( p \lor q \)

• Example of invalid argument form:
  – Premises: \( p \implies q \lor \sim r \) and \( q \implies p \land r \), conclusion: \( p \implies r \)
Valid Argument-Forms

• Modus ponens (method of affirming):
  – Premises: \( p \rightarrow q \) and \( p \), conclusion: \( q \)

• Modus tollens (method of denying):
  – Premises: \( p \rightarrow q \) and \( \neg q \), conclusion: \( \neg p \)

• Disjunctive addition:
  – Premises: \( p \), conclusion: \( p \lor q \)
  – Premises: \( q \), conclusion: \( p \lor q \)

• Conjunctive simplification:
  – Premises: \( p \land q \), conclusion: \( p, q \)
Valid Argument-Forms

• Disjunctive Syllogism:
  – Premises: $p \lor q$ and $\neg q$, conclusion: $p$
  – Premises: $p \lor q$ and $\neg p$, conclusion: $q$

• Hypothetical Syllogism
  – Premises: $p \rightarrow q$ and $q \rightarrow r$, conclusion: $p \rightarrow r$

• Dilemma: proof by division into cases:
  – Premises: $p \lor q$ and $p \rightarrow r$ and $q \rightarrow r$, conclusion: $r$
Complex Deduction

• Premises:
  – If my glasses are on the kitchen table, then I saw them at breakfast
  – I was reading the newspaper in the living room or I was reading the newspaper in the kitchen
  – If I was reading the newspaper in the living room, then my glasses are on the coffee table
  – I did not see my glasses at breakfast
  – If I was reading my book in bed, then my glasses are on the bed table
  – If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table

• Where are the glasses?
Fallacies

• A fallacy is an error in reasoning that results in an invalid argument

• Three common fallacies:
  – Vague or ambiguous premises
  – Begging the question (assuming what is to be proved)
  – Jumping to conclusions without adequate grounds

• Converse Error:
  – Premises: \( p \to q \) and \( q \), conclusion: \( p \)

• Inverse Error:
  – Premises: \( p \to q \) and \( \sim p \), conclusion: \( \sim q \)
Fallacies

• It is possible for a valid argument to have false conclusion and for an invalid argument to have a true conclusion:
  – Premises: if John Lennon was a rock star, then John Lennon had red hair, John Lennon was a rock star; Conclusion: John Lennon had red hair
  – Premises: If New York is a big city, then New York has tall buildings, New York has tall buildings; Conclusion: New York is a big city
Contradiction

- Contradiction rule: if one can show that the supposition that a statement \( p \) is false leads to a contradiction, then \( p \) is true.

- Knight is a person who always says truth, knave is a person who always lies:
  - A says: B is a knight
  - B says: A and I are of opposite types

What are A and B?
Digital Logic Circuits

• Digital Logic Circuit is a basic electronic component of a digital system
• Values of digital signals are 0 or 1 (bits)
• Black Box is specified by the signal input/output table
• Three gates: NOT-gate, AND-gate, OR-gate
• Combinational circuit is a combination of logical gates
• Combinational circuit always correspond to some boolean expression, such that input/output table of a table and a truth table of the expression are identical
Digital Logic Circuits

• A recognizer is a circuit that outputs 1 for exactly one particular combination of input signals and outputs 0’s for all other combinations
• Multiple-input AND and OR gates
• Finding a circuit that corresponds to a given input/output table:
  – Construct equivalent boolean expression using disjunctive normal form: for all outputs of 1 construct a conjunctive form based on the truth table row. All conjunctive forms are united using disjunction
  – Construct a digital logic circuit equivalent to the boolean expression
Digital Logic Circuits

• Design a circuit for the following output: (0, 0, 1, 1, 0, 0, 1, 0)

• Two digital logic circuits are equivalent iff their input/output tables are identical

• Simplification of circuits

• Scheffer stroke (NAND)

• Peirce arrow (NOR)
Number Systems

• Decimal number system
• Binary number system
• Conversion between decimal and binary numbers
• Binary addition and subtraction
Digital Circuits for Addition

• Half Adder – addition of two bits
• Full Adder – addition of two bits and a carry
• Parallel Adder – addition of multi-bit numbers
Negative Numbers

• Two’s complement of a positive integer a relative to a fixed bit length n is the binary representation of $2^n - a$

• To find an 8-bit complement:
  – Write 8-bit binary representation of the number
  – Flip all bits (one’s complement)
  – Add 1 to the obtained binary

• Addition of negative numbers
Hexadecimal Numbers

• Hexadecimal notation is a number system with base 16
• Digits of hexadecimal number system
• Conversion between hexadecimal and binary and hexadecimal and decimal systems