Chapter 4

ENUMERATION

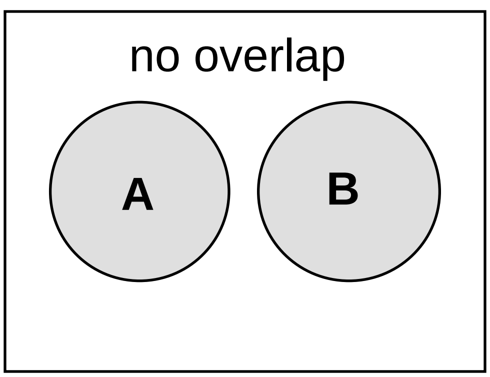
4.1 BASICS OF COUNTING

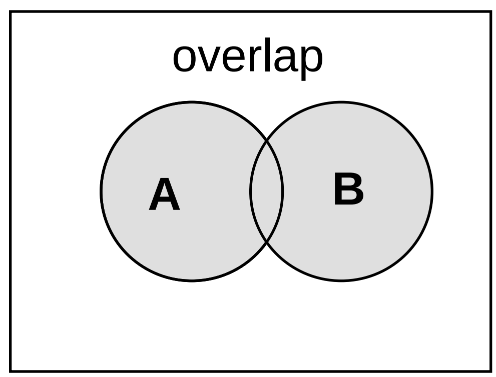
Giving names to the rules of sum, product, and quotient was a significant contribution to the technology of counting. The text preserves the setting of "tasks" in which these rules were first asserted.

RULE of SUM

Rule of Sum: Let A and B be finite sets such that $A \cap B = \emptyset$. Then $|A \cup B| = |A| + |B|$.

Proof: The Rule of Sum concerns counting the union of two sets with no overlap.





Formally, a proof would proceed by induction on the number of elements of B. $\diamondsuit$

Example 4.1.1: Suppose there are 19 Flemish majors, 17 Frisian majors, and no double majors. How many ways are there to choose someone who is either a Flemish major or a Frisian major?

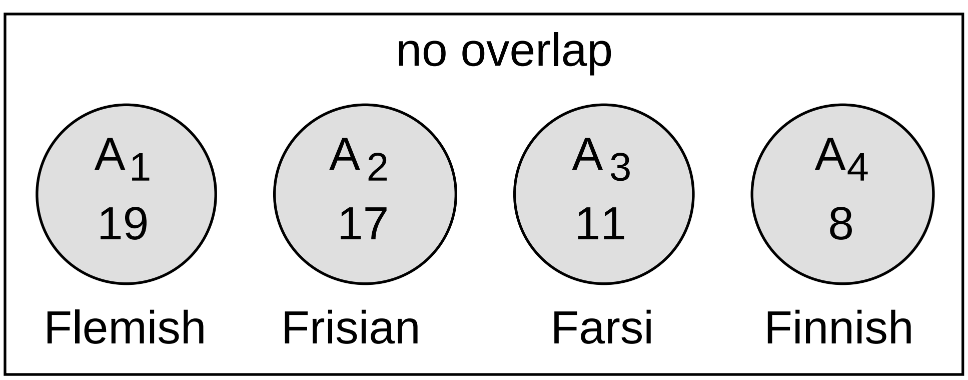
Answer. $19 + 17 = 36$

Iterated Rule of Sum: Let $A_1, A_2, \ldots, A_n$ be pairwise disjoint finite sets. Then

$$\left| \bigcup_{j=1}^{n} A_j \right| = \sum_{j=1}^{n} |A_j|$$

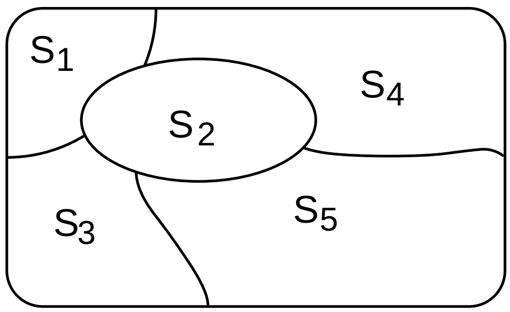
Example 4.1.2: Suppose there are also 11 Farsi majors and 8 Finnish majors, and no multiple majors. Then the total number of students majoring in one of these languages equals

$19 + 17 + 11 + 8 = 55$



PARTITIONS

DEF: A ***partition of a set*** S is a family of nonempty subsets $S_1, S_2, \ldots, S_k$, called ***cells***, such that
(1) their pairwise intersections are empty
(2) their union is S.



Example 4.1.3: The integers are partitioned into odd and even numbers.

Example 4.1.4: The set of logical propositions is partitioned into tautologies, self-contradictions, and contingencies.

Remark: The Iterated Rule of Sum says that the number of objects in the whole equals the sum of the numbers of objects in the parts.

Remark: Applying the Rule of Sum is a matter of partitioning into parts that are easily counted.

Example 4.1.5: The total number of items given you by True Love is

$$\sum_{n=1}^{12}\sum_{j=1}^{n} j \quad = \quad \sum_{n=1}^{12}\sum_{j=1}^{n} j^{\underline{1}}$$

Integral calculus gives this well-known formula.

$$\int_{x=a}^{b} x^n \; = \; \left.\frac{x^{n+1}}{n+1}\right|_{a}^{b}$$

This discrete analogue is provable by induction on the number $b+1-a$ of terms in the sum.

$$\sum_{x=a}^{b} x^{\underline{n}} \; = \; \left.\frac{x^{\underline{n+1}}}{n+1}\right|_{a}^{b+1}$$

It follows that the total number of gifts is

$$\sum_{n=1}^{12} \left.\frac{j^{\underline{2}}}{2}\right|_{j=1}^{n+1} = \frac{1}{2}\sum_{n=1}^{12}(n+1)^{\underline{2}}$$

$$= \frac{1}{2}\sum_{n=2}^{13} n^{\underline{2}} = \frac{1}{2}\cdot\left.\frac{n^{\underline{3}}}{3}\right|_{2}^{14} = \frac{14^{\underline{3}}}{6} = 364$$

RULE of PRODUCT

Rule of Product: Let A and B be finite sets. Then $|A \times B| = |A| \cdot |B|$.

Proof: Use mathematical induction on $|B|$. $\Diamond$

$$\begin{matrix} (a_1, b_1) & (a_1, b_2) & \cdots & (a_1, b_n) \\ (a_2, b_1) & (a_2, b_2) & \cdots & (a_2, b_n) \\ \vdots & \vdots & \ddots & \vdots \\ (a_m, b_1) & (a_m, b_2) & \cdots & (a_m, b_n) \end{matrix}$$

Example 4.1.6: How many ways are there to pick two students with different majors from among the 19 Flemish majors and the 17 Frisian majors?

Answer. $19 \cdot 17 = 323$

Analysis. Count the ordered pairs (w, x) such that w is a Flemish major and x a Frisian major.

Iterated Rule of Product: Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$\left| \prod_{j=1}^{n} A_j \right| = \prod_{j=1}^{n} |A_j|$$

Example 4.1.7: Include the 11 Farsi majors and the 8 Finnish majors, and then pick four students, one representing each language.

New Answer. $19 \cdot 17 \cdot 11 \cdot 8$

Analysis. Now you are counting the ordered quadruples (w, x, y, z) with

$$w \in F_l, x \in F_r, y \in F_a, z \in F_i.$$

Example 4.1.8: How many ways are there to deal a 5-card sequence from a 52-card poker deck?

Answer. $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = 52^{\underline{5}}$

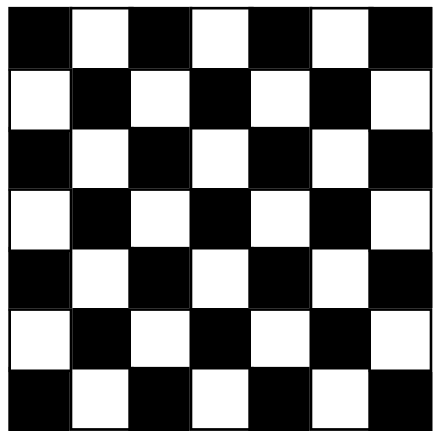
REVIEW §1.8: ***falling power***

$$x^{\underline{r}} = x(x-1)(x-2)\cdots(x-r+1).$$

Remark: In general, falling powers count ordered selections. §4.4 shows how binomial coefficients count subsets (i.e., unordered selections).

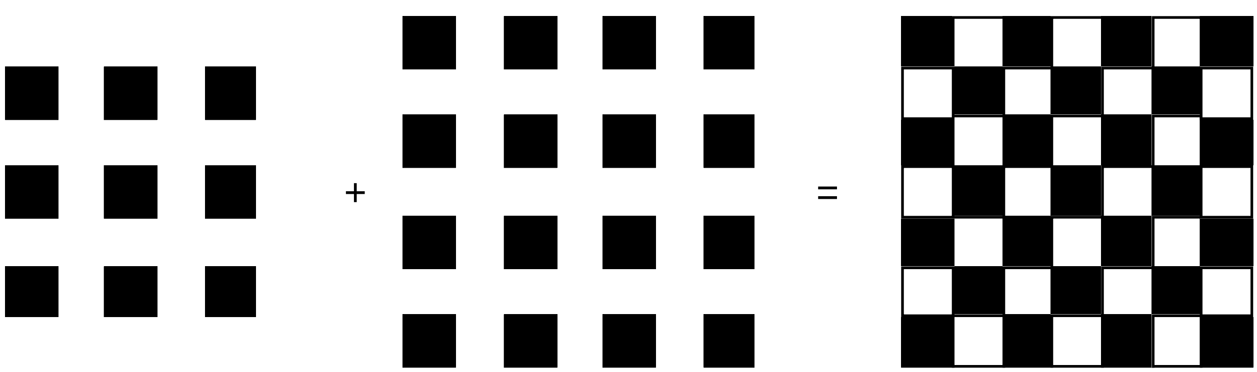
COMBINING RULES of SUM and PRODUCT

Example 4.1.9: How many ways are there to select an ordered pair of numbers from 1 to 7 (repetition allowed) so that the sum is even? This is equivalent to counting the black squares.



Solution. Partition into two cases.

both even	$3 \cdot 3 = 9$	by rule of product
both odd	$4 \cdot 4 = 16$	by rule of product
total	$9 + 16 = 25$	by rule of sum



RULE of DIVISION

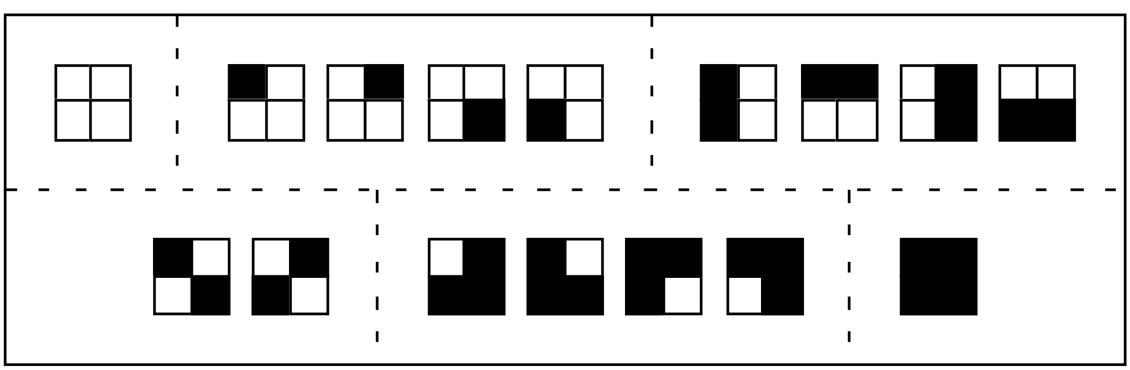
Rule of Division: Let S be a set of n items partitioned into cells of size d. Then the number of cells in the partition is $\frac{n}{d}$.

Example 4.1.10: Suppose that 800 pieces of chocolate candy are to be sold 50 pieces to a box. How many boxes are required?

Answer. $\frac{800}{50} = 16$

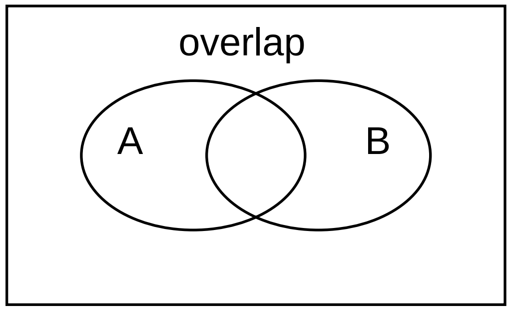
Remark: Counting the number of cells can be much more difficult when different cells do not contain the same number of items.

Example 4.1.11: The 16 ways to 2-color a 2-by-2 checkerboard fall into six equivalence classes under reflection and rotation.



INCLUSION-EXCLUSION

The Rule of Sum can be generalized to count the union of two sets with nonempty overlap.



Rule of Inclusion-Exclusion: Let A and B be finite sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Proof: The sum $|A| + |B|$ counts each object in $A \cap B$ twice. $\diamondsuit$

Example 4.1.12: Count the number of bitstrings of length 8 that have either 1 as a prefix or 00 as a suffix.

Answer. Let A = set of bitstrings with 1 as a prefix, and let B = set of bitstrings with 00 as a suffix. Then

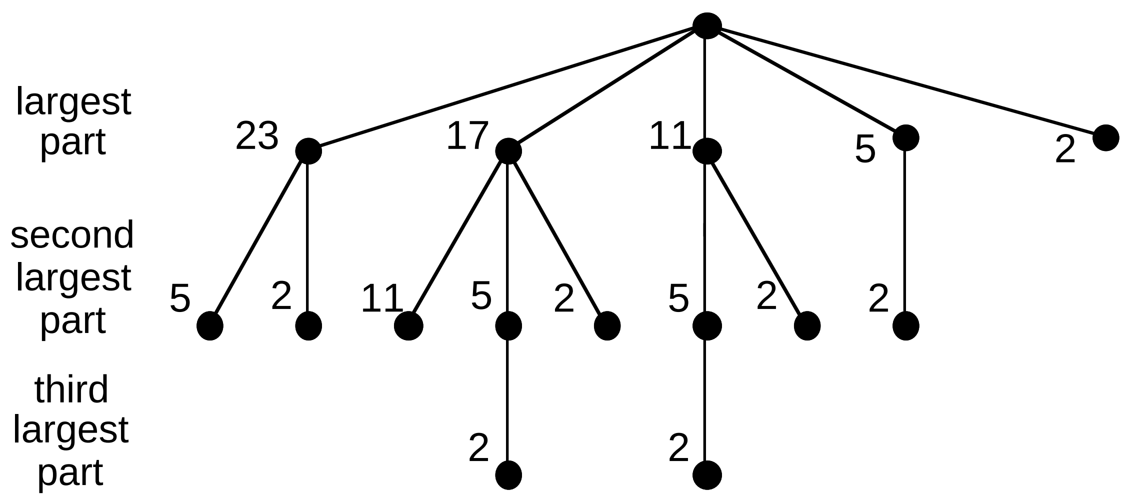
$$|A| = 2^7 \quad |B| = 2^6 \quad |A \cap B| = 2^5$$
$$|A \cup B| = |A| + |B| - |A \cap B| = 2^7 + 2^6 - 2^5$$

TREE DIAGRAMS

A tree diagram can be used in many different to count a set, for instance by illustrating a sequence of successively finer partitions.

DEF: A ***knapsack problem*** is to find in a set of numbers the subset whose value is largest, but not in excess of a fixed limit.

Example 4.1.13: Count the subsets of 2, 5, 11, 17, 23 whose sum is at most 29.



Each vertex represents the subset comprising the numbers encountered on a path to it from the root. Thus, the total number of subsets whose sum is at most 29 is equal to the number of vertices, which is 16.

4.2 PIGEONHOLE PRINCIPLE

Basic Pigeonhole Principle. *If $k+1$ objects are placed into k boxes, then there is at least one box containing two or more objects.*

Set-Theoretic Pigeonhole Principle. *Let $f : D \to R$ be a function with finite domain and codomain. If any two of these three properties holds, then so does the third.*

P_1. f is one-to-one.

P_2. f is onto.

P_3. $|D| = |R|$.

Proof: All three cases are provable by straightforward induction. ♢

TERMINOLOGY: Conceptualization of domain D as a set of pigeons and of codomain R as the set of holes to which they are assigned leads to the name ***pigeonhole principle***.

DIRECT APPLICATIONS

Example 4.2.1:
[one-to-one $\wedge$ same cardinality $\Rightarrow$ onto].

Given a group of 7 persons, suppose no two of them were born on the same day of the week. Then each day of the week was the day on which at least one of them was born.

Example 4.2.2:
[one-to-one $\wedge$ onto $\Rightarrow$ same cardinality].

Given a group of n persons, suppose that no two of them were born the same day of the week, and that each day of the week is the day on which at least one of them were born. Then $n = 7$.

Example 4.2.3:
[same cardinality $\wedge$ onto $\Rightarrow$ one-to-one].

Given a group of 7 persons, suppose that each day of the week is the day on which at least one of them was born. Then no two of them were born the same day of the week.

INDIRECT APPLICATIONS

Example 4.2.4: ***alternative pigeonhole principle***

[more pigeons than pigeonholes $\Rightarrow$ not 1-to-1]

Given $n \geq 8$ persons, there must be two of them who were born on the same day of the week.

Proof: Suppose, to the contrary, that the assignment of days to persons is 1-to-1. Let S be the subset of days of the week on which at least one of the n births occurred. Then, using S as the codomain, the assignment is 1-to-1 and onto. By the pigeonhole principle, $|S| = n$, which contradicts the facts that S is a subset of the seven weekdays and that $n \geq 8$.

Example 4.2.5: A human head never has over 500,000 hairs (the pigeonholes). [Human heads have about 100,000 hairs on average.] The population of NYC is at least 7,000,000 (the pigeons). Therefore, there exist two equally hirsute persons in NYC.

TRICKIER VARIATIONS

Example 4.2.6: In any group of $n \geq 2$ persons (pigeons), at least two of them have the same number of acquaintances (pigeonholes) within the group. (The philosophical construct of self-acquaintance is not counted. Nor is plural acquaintance with someone's psychiatrically multiple personalities.)

Proof: Suppose, to the contrary, that the mapping from pigeons

$$p_1, p_2, \ldots, p_n$$

to pigeonholes is 1-to-1. The possible numbers of acquaintance are

$$0, 1, \ldots, n-1,$$

which is the same as the number of persons. By the pigeonhole principle, this mapping is onto, which implies that someone knows everyone else ($n-1$ acquaintances) and someone else knows no one (0 acquaintances). However, these two occurrences are incompatible with each other.

Example 4.2.7: In your bureau drawer, you have 10 pairs of different socks, all loosely mixed together. In total darkness, you pack your suitcase. How many socks must you pack to be sure that you have at least one pair?

Solution. The 10 sock-types are pigeonholes. Individual socks are the pigeons. Eleven socks forces a pair.

Geenralized Pigeonhole Principle. *If N objects are placed into k boxes, then there is at least one box containing* $\left\lceil \frac{N}{k} \right\rceil$ *or more objects.*

Example 4.2.8: A human head never has over 500,000 hairs (the pigeonholes). The population of NYC is at least 7,000,000 (the pigeons). Therefore, there exist fourteen or more equally hirsute persons in NYC.

4.3 PERMUTATIONS & COMBINATIONS

DEF: An ***ordered* r*-selection*** from a set S ("traditional" name: **r*-permutation***) is a sequence of r objects from S.

NOTATION: The number of ordered r-selections is denoted $P(n, r)$ in many undergraduate level textbooks, perpetuating the era in which it was difficult to typeset its value, which is $n^{\underline{r}}$.

DEF: An ***unordered* r*-selection*** from a set S ("traditional" name: **r*-combination***) is a subset of r objects from S.

NOTATION: The number of unordered r-selections is denoted $C(n, r)$ in many undergraduate level textbooks, perpetuating the era in which it was difficult to typeset the binomial coefficient

$$\binom{n}{r}$$

DISAMBIGUATION: The primary mathematical meaning of ***permutation*** is a bijection from a set onto itself.

COUNTING ORDERED SELECTIONS

REVIEW EXAMPLE 4.1.8: Ordered selections were introduced as a mathematical model for the number of different possible 5-card sequences from a 52-card poker deck.

Example 4.3.1: Suppose there are 7 flags, all of different colors. How many different signals can be formed by running three flags to the top of a flagpole?

Answer. $7^{\underline{3}} = 7 \cdot 6 \cdot 5$ $\diamondsuit$

Proposition 4.3.1. $P(n, r) = n^{\underline{r}}$. *That is, the number of ordered selections of* r *items (without repetition) from a set of* n *items equals* $n^{\underline{r}}$.

Proof: By induction on r, using the Rule of Product. $\diamondsuit$

CLASSROOM EXERCISE. Suppose that repetitions are allowed. Then how many ways can you choose an ordered list of r items from a set of n items?

ANSWER:

COUNTING UNORDERED SELECTIONS

Proposition 4.3.2. $C(n,r) = \dfrac{n^{\underline{r}}}{r!}$.

Proof: Let ${}_rP_n$ be the set of all ordered r-selections from a set S of size n. Prop 4.3.1 establishes that ${}_rP_n$ contains $n^{\underline{r}}$ items.

Consider a partition of the set ${}_rP_n$ such that two ordered r-selections are in the same cell if and only if they differ only by the order of the objects from S. Then the cells of this partition are in bijective correspondence with the set ${}_rC_n$ of r-combinations of objects from S.

Each cell of this partition contains exactly $r!$ ordered r-selections. By the rule of division, it follows that $C(n,r) = \dfrac{n^{\underline{r}}}{r!}$. $\diamondsuit$

Example 4.3.2: The number of unordered selections of two cards from 13 equals

$$C(13,2) = \frac{13^{\underline{2}}}{2!} = \frac{13 \cdot 12}{2 \cdot 1} = 78$$

Example 4.3.3: The number of 5-card draw poker hands in a 52-card deck equals

$$C(52,5) = \frac{52^{\underline{5}}}{5!}$$

Example 4.3.4: The number of different possible flushes in a 5-card poker hand equals

$$4 \cdot C(13,5)$$

Example 4.3.5: The number of different hands that constitute a full house equals

$$13 \cdot C(4,3) \cdot 12 \cdot C(4,2)$$

Example 4.3.6: The number of different hands that contain two pair equals

$$C(13,2) \cdot C(4,2) \cdot C(4,2) \cdot C(44,1)$$

4.4 BINOMIAL COEFFICIENTS

DEF: The ***binomial coefficient*** $\binom{n}{r}$ is the coefficent of x^r in the expansion of the (exponentiated) binomial $(1+x)^r$ to a polynomial.

NOTATION: for the binomial coefficient $\begin{pmatrix} n \\ r \end{pmatrix}$

$$\begin{aligned}(1+x)^n &= \sum_{r=0}^{n} \binom{n}{r} x^r \\ &= \binom{n}{0}x^0 + \binom{n}{1}x^1 + \cdots + \binom{n}{n}x^n\end{aligned}$$

REVIEW FROM PRE-COLUMBIA:

$$(1+x)^0 = 1$$

$$(1+x)^1 = 1 + x$$

$$(1+x)^2 = 1 + 2x + x^2$$

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3$$

$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

$$(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$

Theorem 4.4.1 [Binomial Theorem].

$$\binom{n}{r} = C(n,r) = \frac{n^{\underline{r}}}{r!}$$

That is, in the expansion of $(1 + x)^n$, *the coefficient of* x^r *equals* $C(n,r)$.

Proof: The coefficient of x^r in the expansion

$$(1+x)^n = \overbrace{(1+x)(1+x)\cdots(1+x)}^{n \text{ factors}}$$

(usually denoted $\binom{n}{r}$) equals the number $C(n,r)$ of ways to select from the total set of n factors a subset of exactly r factors in which x is the designated term. By Prop 4.4.0, $C(n,r) = \frac{n^{\underline{r}}}{r!}$ ♢

Remark: When n is not an integer, the binomial theorem remains true if you use the formula $\binom{n}{r} = \frac{n^{\underline{r}}}{r!}$, but not if you use $\frac{n!}{(n-r)!r!}$.

SYMMETRY IDENTITY

Theorem 4.4.2. *Symmetry.*

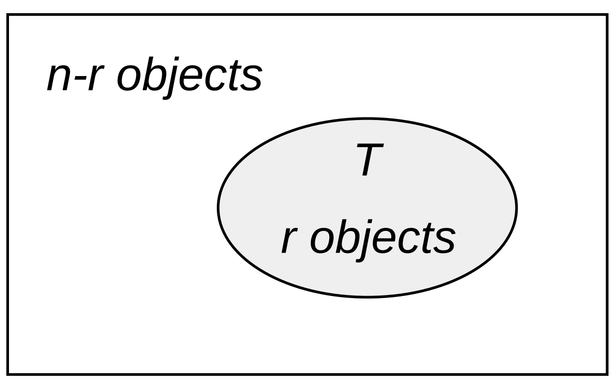
$$\binom{n}{r} = \binom{n}{n-r}$$

Proof: (1st proof: computational)

$$\binom{n}{r} = \frac{n^{\underline{r}}}{r!} = \frac{n!/(n-r)!}{r!} = \frac{n!/r!}{(n-r)!}$$

$$= \frac{n^{\underline{n-r}}}{(n-r)!} = \binom{n}{n-r}$$

Proof: (2nd proof: combinatorial)
The number of ways to select a subset T of r objects from a set S of n objects equals the number of ways to select the complementary set $S - T$ of $n - r$ objects that are to be excluded. $\diamondsuit$



PASCAL'S RECURSION

Theorem 4.4.3. *Pascal's Recursion*

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

Proof: (1st proof: computational)

$$\binom{n-1}{r-1} + \binom{n-1}{r} = \frac{(n-1)!}{(n-r)!(r-1)!} + \frac{(n-1)!}{(n-r-1)!r!}$$

$$= \frac{(n-1)!r}{(n-r)!r!} + \frac{(n-1)!(n-r)}{(n-r)!r!}$$

$$= \frac{(n-1)![r+(n-r)]}{(n-r)!r!} = \frac{(n-1)!n}{(n-r)!r!}$$

$$= \frac{n!}{(n-r)!r!} = \binom{n}{r}$$

Proof: (2nd proof: combinatorial)
Let $\mathcal{F}$ be the family of all r-subsets of a set S of n objects, so that $|\mathcal{F}| = \binom{n}{r}$

Then $\mathcal{F}$ can be partitioned into two subfamilies:

$$\mathcal{F}_1 = \{T \subseteq S \mid (\text{object } n) \in T\} \qquad |\mathcal{F}_1| = \binom{n-1}{r-1}$$

$$\mathcal{F}_2 = \{T \subseteq S \mid (\text{object } n) \notin T\} \qquad |\mathcal{F}_2| = \binom{n-1}{r}$$

◊

Corollary 4.4.4. *The Pascal recurrence system*

$$p_{0,r} = \begin{cases} 1 & \textit{if } r = 0 \\ 0 & \textit{else} \end{cases} \qquad \textit{(row 0)}$$

$$p_{n,0} = 1 \qquad \textit{(column 0)}$$

$$p_{n,r} = p_{n-1,r-1} + p_{n-1,r}$$

yields the values $p_{n,r} = \binom{n}{r}$. ◊

The ***Pascal triangle*** with entry $p_{n,r} = \binom{n}{r}$ is calculated recursively, row by row, with the aid of the Pascal recurrence system:

	r = 0	1	2	3	4	5	6
n = 0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

Often Pascal's triangle appears this way:

```
n = 0                       1
  1                    1         1
  2               1         2         1
  3           1        3         3        1
  4        1       4        6         4       1
  5     1      5       10        10       5      1
  6   1     6      15       20        15      6     1
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BINOMIAL COEFFICIENT IDENTITIES

Theorem 4.4.5. *Subset size sum.*

$$\sum_{r=0}^{n} \binom{n}{r} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

Proof: (1st proof: computational) By induction on n, using Pascal's recursion. $\diamondsuit$

Proof: (2nd proof: combinatorial) Both sides count all the subsets. $\diamondsuit$

Proof: (3rd proof: corollary to binomial thm) Expand $(x+y)^n$ with $x = y = 1$. $\diamondsuit$

Theorem 4.4.6. *Vandermonde Identity Let m, n, and r be nonnegative integers such that $r \leq m$ and $r \leq n$. Then*

$$\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}$$

Proof: Partition a set S of size $m+n$ into subsets T and U of sizes m and n. To choose r objects from S, one may choose k objects from T and the remaining $r-k$ objects from U. $\diamondsuit$

Theorem 4.4.7. *Alt sum of binom coeffs.*

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} \;=\; 0$$

Proof: Expand $(x+y)^n$ with $x = 1$ and $y = -1$. $\diamondsuit$

Theorem 4.4.8. Σ *squares of binom coeffs.*

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Proof: Apply symmetry, then Vandermonde.

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k} = \binom{2n}{n} \qquad \diamondsuit$$

4.5 GENERALIZED PERMUTATIONS AND COMBINATIONS

Some counting problems involve repetition.

SEQUENCES with UNRESTRICTED REPETITION

Prop 4.5.1. *The number of sequences of length r selected from a set of n objects is n^r.*

Proof: Rule of Product. ♢

Example 4.5.1: There are 26^4 four-letter alphabetic strings in the English alphabet: $AAAA, AAAB, \ldots, ZZZZ$.

Example 4.5.2: There are 10^3 (unsigned) base-ten numerals with three or fewer digits: 0 (means 000), 1 (means 001), ..., 999.

Example 4.5.3: Most license plates in New Jersey are formed by an ordered pair of 3-strings whose characters are either letters or digits. There are 36^6 such pairs.

SEQUENCES with RESTRICTED REPETITION

Prop 4.5.2. *Let $S = \{a_1, a_2, \ldots, a_k\}$, and let $n = n_1 + n_2 + \cdots + n_k$. The number of length-$n$ sequences in S with n_j occurrences of object a_j, for $j = 1, \ldots, k$ is*

$$\frac{n!}{n_1! n_2! \cdots n_k!}$$

Proof: Rule of Quotient. ♢

Example 4.5.4: How many different strings can be formed by rearranging the letters of the word BANANA?
Solution: 3A, 1B, 2N.

$$\frac{6!}{3!1!2!} = 60$$

Example 4.5.5: How many different strings can be formed by rearranging the letters of the word BOOKKEEPER?
Solution: 1B, 3E, 2K, 2oh, 1P, 1R.

$$\frac{10!}{1!3!2!2!1!1!}$$

COMBINATIONS with UNRESTRICTED REPETITION

Example 4.5.6: Three dice are rolled. Count the number of possible combinations of values.

NOTATION: Represent a dice outcome as a non-descending 3-sequence in the set $\{1, 2, \ldots, 6\}$.

Method 1. One at a time.

111	112	113	114	115	116				
122	123	124	125	126	133	134	135	136	
144	145	146	155	156	166				
222	223	224	225	226	233	234	235	236	
244	245	246	255	256	266				
333	334	335	336	344	345	346	355	356	366
444	445	446	455	456	466	555	556	566	666

$7 \cdot 6 \ + \ 6 \ + \ 8 = \mathbf{56}$

Method 2. By partition structure.

$$\begin{cases} \binom{6}{1} = 6 & \text{3-of-a-kind xxx} \\ \binom{6}{1}\binom{5}{1} = 30 & \text{one-pair xxy} \\ \binom{6}{3} = 20 & \text{all different xyz} \end{cases}$$

$6 + 30 + 20 = \mathbf{56}$

Method 3. By an ingenious bijection, which is a formal name for pigeonholing.

Domain D: nondescending triples in $\{1, ..., 6\}$.

Codomain R: 8-bit strings with exactly three 0's

Strategy

Domain D models the set to be counted.
Codomain R is easy to count.
Construct a bijection $f : D \to R$, and count R.

Fact 1. *The set of triples* $[t_1, t_2, t_3]$ *correspond bijectively to the set of 6-tuples* $(k_1, k_2, \ldots, k_6)$ *such that*

$$k_j = \#\text{occurrences of } j \text{ in } [t_1, t_2, t_3]$$

and $\sum_{j=1}^{6} k_j = 3$.

Proof: Each such 6-tuple is the image of a unique triple. $\diamondsuit$

Example 4.5.7: $[2, 2, 4] \mapsto (0, 2, 0, 1, 0, 0)$.

Fact 2. *The set of 6-tuples* $(k_1, k_2, \ldots, k_6)$ *as in Fact 1 correspond bijectively to the set of length-8 bitstrings with 5 ones and 3 zeroes.*

Proof: This algorithm is the bijection:

Algorithm 4.5.1: tuples to bitstrings

Input: a 6-tuple $(k_1, k_2, \ldots, k_6)$ as in Fact 1
Output: a bitstring of length $6 + 3 - 1$,
with 3 zeroes

Initialize output string $s := \lambda$

For $j := 1$ **to** 5
  For $i := 1$ **to** k_j $\{s := s \cdot 0\}$
  $s := s \cdot 1$
  continue with next j

For $i := 1$ **to** k_6 $\{s := s \cdot 0\}$

Example 4.5.8:
$[2, 2, 4] \mapsto (0, 2, 0, 1, 0, 0) \mapsto 10011011.$

Conclusion: The number of possible outcomes of a roll of three indistinguishable dice is $\binom{8}{3}$

Theorem 4.5.3. *The number of ways to choose r objects from a set of cardinality n, if unrestricted repetitions are allowed, is*

$$\binom{n+r-1}{r}$$

Proof: Generalize the example above. ◊

CLASSROOM EXERCISE

How many possible combinations of seven coins can be formed from U.S. coins presently in circulation? 1, 5, 10, 25, 50, 100

ANSWER:

4.9 PARTITIONS

The set $\{1, 2, 3, 4\}$ has 16 subsets.

∅						$\binom{4}{0}$
1	2	3	4			$\binom{4}{1}$
12	13	14	23	24	34	$\binom{4}{2}$
123	124	134	234			$\binom{4}{3}$
1234						$\binom{4}{4}$
						$2^4 = 16$

The set $\{1, 2, 3, 4\}$ has 15 partitions, as follows:

1234

$12|34 \quad 13|24 \quad 14|23 \quad 1|234 \quad 2|134 \quad 3|124 \quad 4|123$

$1|2|34 \quad 1|3|24 \quad 1|4|23 \quad 2|3|14 \quad 2|4|13 \quad 3|4|12$

$1|2|3|4$

The types of these partitions are

4, 22, 13, 112, and 1111

corresponding to the partitions of the number 4.

Our objective is to count the partitions of a set of n objects without listing them all.

DEF: The ***Stirling subset number***

$$\left\{ {n \atop r} \right\}$$

is the number of partitions of n objects into r unlabeled, nonempty cells. It is also called the ***Stirling coefficient of the second kind***.

Example 4.9.1: From the itemization above,

$$\left\{ {4 \atop 1} \right\} = 1 \quad \left\{ {4 \atop 2} \right\} = 7 \quad \left\{ {4 \atop 3} \right\} = 6 \quad \left\{ {4 \atop 4} \right\} = 1$$

Theorem 4.9.1 [Stirling's Recursion].

$$\left\{ {n \atop r} \right\} = \left\{ {n-1 \atop r-1} \right\} + r \left\{ {n-1 \atop r} \right\}$$

Proof: The nth object is isolated in a cell by itself in $\left\{ {n-1 \atop r-1} \right\}$ partitions. Each of the remaining partitions is formed by first partitioning the $n-1$ other objects into r nonempty cells and then selecting one of them as a cell for the nth object. Bythe rule of product, there are $r \left\{ {n-1 \atop r} \right\}$ ways to do this. The rule of sum now implies the conclusion. ♢

Remark: Although there is no elementary formula for Stirling subset numbers, Stirling's recursion facilitates the construction of a triangular array, reminiscent of the Pascal triangle for binomial coefficients.

$$\left\{ \begin{matrix} n \\ r \end{matrix} \right\} = \left\{ \begin{matrix} n-1 \\ r-1 \end{matrix} \right\} + r \left\{ \begin{matrix} n-1 \\ r \end{matrix} \right\}$$

$\left\{ \begin{matrix} n \\ r \end{matrix} \right\}$	r = 0	1	2	3	4	5
n = 0	1	0	0	0	0	0
1	0	1				
2	0	1	1			
3	0	1	3	1		
4	0	1	7	6	1	
5	0	1	15	25	10	1