7) Prove \( P(n) \): \( 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \) for \( n \geq 1 \)

Base case: \( P(1) \):

\[ P(1): 1^2 = \frac{1(1+1)(2\cdot1+1)}{6} \]
\[ \Rightarrow 1 = \frac{6}{6} = 1 \checkmark \]

Assume: \( P(k) \): for \( k \geq 1 \):

\[ 1^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6} \]

Prove: \( P(k+1) \):

\[ 1^2 + \ldots + (k+1)^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6} \]

Use the assumption \( P(k) \) and add \( (k+1)^2 \) to both sides:

\[ 1^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \]
\[ \Rightarrow 1^2 + \ldots + (k+1)^2 = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \]
\[ \Rightarrow 1^2 + \ldots + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6} \]
\[ \Rightarrow 1^2 + \ldots + (k+1)^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6} \checkmark \]

\( \therefore P(n) \) is true for \( n \geq 1 \).
10. Prove by induction:

\[ P(n) : \quad 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1 \quad \text{for } n \neq 1 \]

**Base case:** Prove \( P(1) \):

\[ 1 \cdot 1! = (1+1)! - 1 \]

\[ \Rightarrow \quad 1 = 2 \cdot 1 - 1 = 1 \quad \checkmark \]

**Assume:** \( P(k) \): for \( n \neq 1 \):

\[ 1 \cdot 1! + \cdots + k \cdot k! = (k+1)! - 1 \quad \text{is true} \]

**Prove:** \( P(k+1) \):

\[ 1 \cdot 1! + \cdots + (k+1)(k+1)! = (k+2)! - 1 \]

Add \( (k+1)(k+1)! \) to both sides of \( P(k) \):

\[ \Rightarrow 1 \cdot 1! + \cdots + k \cdot k! + (k+1)(k+1)! = (k+1)! - 1 + (k+1)(k+1)! \]

\[ = (k+1)! \left[ 1 + (k+1) \right] - 1 \]

\[ = (k+1)! \cdot (k+2) - 1 \]

\[ = (k+2)! - 1 \quad \checkmark \]

\[ \therefore P(n) \text{ is true for } n \neq 1 \]
13. Prove \( p(n) : 2^n > n^2 \) for \( n \geq 4 \) by induction.

**Base case:** Prove \( p(5) \):

\[ 2^5 > 5^2 \]
\[ 32 > 25 \]
\[ \checkmark \]

**Assume:** \( p(k) : 2^k > k^2 \) for \( k \geq 5 \)

**Prove:** \( p(k+1) : 2^{k+1} > (k+1)^2 \)

From the assumption \( p(k) \), multiply both sides by 2:

\[ 2 \Rightarrow 2\cdot 2^k > 2\cdot k^2 \]
\[ \Rightarrow 2^{k+1} > 2\cdot k^2 \]

We can now show that \( 2^{k^2} > (k+1)^2 \) for \( k \geq 5 \) by induction or some construction.

*Clearly, \((k)(k-2) > 1 \) for \( k \geq 5 \) (can be shown inductively but skipped here)*

\[ \Rightarrow k^2 - 2k > 1 \]
\[ \Rightarrow k^2 > 2k + 1 \]
\[ \Rightarrow k^2 + k^2 > k^2 + 2k + 1 \]
\[ \Rightarrow 2k^2 > (k+1)^2 \]

Thus,

\[ 2^{k+1} > 2k^2 > (k+1)^2 \]
\[ \Rightarrow 2^{k+1} > (k+1)! \]
\[ \checkmark \]

\[ \therefore \ p(n) \text{ is true for } n \geq 4 \]
The number of breaks needed is \( n-1 \), which we can prove by strong induction.

**Base case:** for \( n = 2 \), we need to do 1 break, and \( n-1 = 1 \) \( \checkmark \)

**Assume:** for \( k \geq 2 \), \( n \), that \( P(n) \) is true. That is, a bar of \( n \) squares requires \( n-1 \) breaks.

**Prove:** \( P(k+1) \) is true.

A bar with \( k+1 \) squares can be broken into two bars with \( b \) squares and \( c \) squares such that \( b+c = k+1 \).

The bar of size \( b \) takes \( b-1 \) breaks, and the bar of size \( c \) takes \( c-1 \) breaks, and breaking the "\( k+1 \)" bar in two takes 1 break

\[
\Rightarrow (b-1) + (c-1) + 1 = \text{total \# breaks}
\]

\[
\Rightarrow (b+c)-1 = \text{\"\"\"\"
\]

\[
\Rightarrow (k+1)-1 = k = \text{total \# breaks}
\]

\[
\Rightarrow P(k+1) \text{ is true by S.M.I.}
\]

\[
\Rightarrow P(n) \text{ is true for } n \geq 1.
\]

**51.** \( P(1) \rightarrow P(2) \) is invalid.
Problem 1:

Prove: \( P(h) \): a complete/full binary tree of height \( h \) has \( 2^h \) leaves for \( h \geq 0 \)

**Base case:** for \( h = 0 \), the tree has one node

\[ \Rightarrow 1 \text{ leaf, and } 2^0 = 1 \]

\[ \Rightarrow P(0) \text{ is true} \]

Assume: for \( k \geq 0 \) \( P(k) \) is true

Prove: \( P(k+1) \)

Since a tree of height \( k \) has \( 2^k \) leaves from assumption of \( P(k) \), then each node at height \( k \) would have 2 children for a tree of height \( k+1 \)

\[ \Rightarrow \text{The number of leaves } = 2 \cdot \text{Number of nodes at height } k \]

\[ = 2 \cdot \# \text{ leaves in tree of height } k \]

\[ = 2 \cdot 2^k \]

\[ = 2^{k+1} \]

\[ \Rightarrow P(k+1) \text{ is true} \]

\[ \Rightarrow P(h) \text{ true for } h \geq 0. \]
Problem 2:
You can prove this almost identically to Problem 1.
or, we can quickly prove this summation formula via induction:

\[ p(h) : 1 + 2 + 4 + \cdots + 2^h = 2^{h+1} - 1 \quad \text{for } h \geq 0 \]

Base case: \( h = 0 \):

\[ 2^0 = 2^{0+1} - 1 \]
\[ \Rightarrow 1 = 2 - 1 \]
\[ = 1 \checkmark \]
\[ \Rightarrow p(0) \text{ is true} \]

Assume: for \( n = k \geq 0 \), \( p(k) \) is true

\[ \Rightarrow 1 + 2 + \cdots + 2^k = 2^{k+1} - 1 \]

Prove: \( p(k+1) \) is true.
Add \( 2^{k+1} \) to both sides of \( p(k) \)

\[ \Rightarrow 1 + 2 + \cdots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1 \]
\[ = 2 \cdot 2^{k+1} - 1 \]
\[ = 2^{k+2} - 1 \]
\[ \Rightarrow p(k+1) \text{ true} \]

\[ \therefore p(h) \text{ true for } h \geq 0 \]