22a) Prove by contrapositive proof. If \( n \) is an integer and \( 3n+2 \) is even, then \( n \) is even.

Let \( p = \text{"n is an integer and } 3n+2 \text{ even"} \)
\( q = \text{"n is even"} \).

Then, proving \( p \Rightarrow q \) is equivalent to proving \( \neg q \Rightarrow \neg p \) on:
"If \( n \) is odd then \( 3n+2 \) is an odd integer"

So, assume \( n \) is odd
\[ \Rightarrow n = 2 \cdot k + 1 \text{ for some } k \in \mathbb{Z} \]
\[ \Rightarrow 3 \cdot n = 3(2 \cdot k + 1) \]
\[ = 6k + 3 \]
\[ \Rightarrow 3 \cdot n + 2 = 6k + 3 + 2 \]
\[ = 6k + 5 \]
\[ = 2(3k + 2) + 1 \]

Let \( l = 3k + 2 \in \mathbb{Z} \)
\[ \Rightarrow 3 \cdot n + 2 = 2 \cdot l + 1, \ l \in \mathbb{Z} \]
\[ \Rightarrow 3n + 2 \text{ is an odd integer} \]

Since \( \neg q \Rightarrow \neg p \), then \( p \Rightarrow q \) \( \checkmark \)
b) Prove by contradiction, which means we assume \( p \land q \) or we assume:

"3n+2 is an even integer and n is odd"

If n is odd
\[ \Rightarrow n = 2k + 1, \; k \in \mathbb{Z} \]
\[ \Rightarrow 3n + 2 = 6k + 5 \\
= 2(3k+2) + 1 \]
Let \( l = 3k + 2 \in \mathbb{Z} \)
\[ \Rightarrow 3n + 2 = 2l + 1, \; l \in \mathbb{Z} \]
\[ \Rightarrow 3n + 2 \text{ is odd, which is a contradiction to the assumption that } 3n + 2 \text{ is even.} \]
\[ \Rightarrow p \land q \text{ is false as } p \Rightarrow q \text{ must be true.} \]

\[ \Rightarrow \]

c) If we try to prove \( p \Rightarrow q \) directly, we would assume \( 3n + 2 \) is even to show n even.

So, assume \( 3n + 2 = 2k, \; k \in \mathbb{Z} \)
\[ \Rightarrow 3n = 2k - 2 \\
\Rightarrow n = \frac{2k - 2}{3} \]

At this point, showing \( n \) is even is not straightforward and is more involved than proofs a) and b).
1b. We are being asked to prove or disprove

\( P(A) = P(B) \implies A = B \)

i) Since \( A \in P(A) \), then \( A \in P(B) \)

\( \exists A \subseteq B \)

Also, \( B \in P(B) \), then \( B \in P(A) \)

\( \exists B \subseteq A \)

Therefore, \( A = B \).

ii) Also, we could do a contrapositive proof:

\( A \neq B \implies P(A) \neq P(B) \).

If, \( A \neq B \), \( \exists x \in A \) such that \( x \notin B \)

\( \exists \exists x \in P(A) \) and \( \exists \exists x \notin P(B) \)

\( \implies P(A) \neq P(B) \).

Therefore since \( A \neq B \implies P(A) \neq P(B) \),

\( P(A) = P(B) \implies A = B \) by contrapositive proof.
19. \(A = \{a, b, c, d\}^3, \quad B = \{y, z\}^3\)
   
a) \(A \times B = \{(a, y), (b, y), (c, y), (d, y), (a, z), (b, z), (c, z), (d, z)\}^3\)
   
b) \(B \times A = \{(y, a), (y, b), (y, c), (y, d), (z, a), (z, b), (z, c), (z, d)\}^3\)

\[\text{Section 1.7}\]

10. \(A - B = \{1, 5, 7, 8\}^3\)
   \(B - A = \{2, 10\}^3\)
   \(A \cap B = \{3, 6, 9\}^3\)

\(\Rightarrow A = \{1, 3, 5, 6, 7, 8, 9\}^3\)
   \(B = \{2, 3, 6, 9, 10\}^3\)

14.e) Show \((B - A) \cup (C - A) = (B \cup C) - A\) by direct proof.

i) So, first show \((B - A) \cup (C - A) \subseteq (B \cup C) - A\).
   Assume \(x \in (B - A) \cup (C - A)\).
   We want to then show \(x \in (B \cup C) - A\).
   \(\Rightarrow x \in (B - A) \text{ OR } x \in (C - A)\)
   \(\Rightarrow (x \in B \text{ AND } x \notin A) \text{ OR } (x \in C \text{ AND } x \notin A)\)
   \(\Rightarrow (x \in B \text{ OR } x \in C) \text{ AND } x \notin A\)
   \(\Rightarrow x \in (B \cup C) \text{ and } x \notin A\)
   \(\Rightarrow x \in (B \cup C) - A\)

Therefore, \((B - A) \cup (C - A) \subseteq (B \cup C) - A\)
(ii) Now show \( (B \cup C) - A \subseteq (B - A) \cup (C - A) \)

Assume \( x \in (B \cup C) - A \)

\( \Rightarrow x \in (B \cup C) \) and \( x \notin A \)

\( \Rightarrow (x \in B \text{ or } x \in C) \) AND \( x \notin A \)

\( \Rightarrow x \in B \text{ and } x \notin A \) or \( x \in C \text{ and } x \notin A \)

\( \Rightarrow x \in B - A \) or \( x \in C - A \)

\( \Rightarrow x \in (B - A) \cup (C - A) \)

\( \Rightarrow (B \cup C) - A \subseteq (B - A) \cup (C - A) \)

So, since \( (B \cup C) - A \subseteq (B - A) \cup (C - A) \)

and \( (B - A) \cup (C - A) \subseteq (B \cup C) - A \)

\( \Rightarrow (B \cup C) - A = (B - A) \cup (C - A) \)

We can also prove the above with set identities:

\( (B \cup C) - A = (B \cup C) \cap A^c \) by difference rule

\( = (B \cap A^c) \cup (C \cap A^c) \) by distributive and associative rules

\( = (B - A) \cup (C - A) \) by difference
20a) \[ A \cap (B \cup C) \]

\[
\begin{align*}
\| & = B \cup C \\
\# & = A \cap (B \cup C)
\end{align*}
\]

22a) No.

Let \( C = \emptyset \) and \( A = \{1, 2, 3\}, B = \emptyset \)

The \( A \cup C = \{1, 2, 3\} \)

\( B \cup C = \{1, 2, 3\} \)

and \( A \neq B \)

You can also see that \( A \cup C = B \cup C \) meaning that for some \( x \), \( x \in (A \cup C) \) AND \( x \in (B \cup C) \)

\[ \Rightarrow x \in A \text{ or } x \in C \text{ AND } x \in B \text{ or } x \in C \]

\[ \Rightarrow (x \in A \text{ and } x \in B) \text{ or } x \in C \]

\[ \Rightarrow x \in (A \cap B) \text{ OR } x \in C \]

So, if \( A \cap B = \emptyset \), \( x \) must be in \( C \) as per our counterexample above.
22b) No.

Let \( C = \emptyset \), \( A = \{1, 3\} \), \( B = \{2, 3\} \)

Then \( A \cap C = B \cap C \) and \( A \neq B \).

**Section 1.8**

18c) To show if \( f(x): \mathbb{R} \to \mathbb{R} \), \( f(x) = \frac{(x+1)}{(x+2)} \) is bijective, we must show \( f(x) \) is 1-1 and onto.

i) Show \( f(x) \) is 1-1:

Assume \( f(x_1) = f(x_2) \) for \( x_1, x_2 \in \mathbb{R} \)

\[
\Rightarrow \frac{x_1 + 1}{x_1 + 2} = \frac{x_2 + 1}{x_2 + 2}
\]

\[
\Rightarrow (x_1 + 1)(x_2 + 2) = (x_2 + 1)(x_1 + 2)
\]

\[
\Rightarrow x_1 x_2 + x_2 + 2x_1 + 2 = x_1 x_2 + x_1 + 2x_2 + 2
\]

\[
\Rightarrow x_2 + 2x_1 = x_1 + 2x_2
\]

\[
\Rightarrow x_1 = x_2
\]

So, \( f(x) \) is 1-1

ii) Show \( f(x) \) is onto:

Let \( f(x) \in \mathbb{R} \)

Then \( f(x) = \frac{(x+1)}{(x+2)} \)

\[
\Rightarrow x \cdot f(x) + 2 f(x) = x + 1
\]

\[
\Rightarrow x \cdot f(x) - x = 1 - 2 f(x)
\]

\[
\Rightarrow x = \frac{(1 - 2 f(x))}{f(x) - 1} \in \mathbb{R}
\]
This would look like \( f(x) \) is onto, but note that \( f(x) \) can be any real number, implying \( f(x) \) should be able to be equal to 1.

But \( f(x) = 1 \Rightarrow x \) undefined, so \( f(x) \) is not onto.

\[ \Rightarrow f(x) \text{ 1-1, not onto} \]
\[ \Rightarrow f(x) \text{ not bijective.} \]

28. \( f(x) = x^2 + 1 \)
\[ g(x) = x + 2 \]

\[ f \circ g = f(g(x)) = f(x+2) = (x+2)^2 + 1 = x^2 + 4x + 5 \]

\[ g \circ f = g(f(x)) = g(x^2 + 1) = x^2 + 3 \]
17. Prove $\sqrt[3]{3}$ is irrational.

As in class, we will do a proof by contradiction. So, assume $\sqrt[3]{3}$ is rational.

$\Rightarrow \sqrt[3]{3} = \frac{p}{q}, \ p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$

$\Rightarrow 3 = \frac{p^3}{q^3}$

$\Rightarrow 3q^3 = p^3$

$\Rightarrow p^3$ is a multiple of 3, which is only possible if $p$ is a multiple of 3, so let $p = 3 \cdot m, \ m \in \mathbb{Z}$

$\Rightarrow 3q^3 = 27m^3$

$\Rightarrow q^3 = 9m^3$

$\Rightarrow 3 | q^3$, which means $3 | q$

Therefore, $3 | p$ and $3 | q \Rightarrow \gcd(p, q) \neq 1$, which is a contradiction.

$\Rightarrow \sqrt[3]{3}$ cannot be rational.

$\Rightarrow \sqrt[3]{3}$ is irrational.
29. Assume there exist finitely many primes of the form \( 6k + 5 \) (ex: 5, 11, 17, 23, etc) for \( k \in \mathbb{Z}^+ \).

Then, let \( q_1, q_2, \ldots, q_n \) be the \( n \) primes of the form \( 6k + 5 \) and let

\[
Q = 6q_1 q_2 \cdots q_n - 1 \quad \text{and note that}
\]

\[
Q = 6(q_1 q_2 \cdots q_{n-1}) + 5 = 6k + 5 \quad \text{for } k = q_1 q_2 \cdots q_{n-1} - 1
\]

Since \( Q \) is of the form \( 6k + 5 \), it is either prime or composite. If it is prime, then \( Q > q_n \), and we have shown a larger \( 6k + 5 \) prime exists, a contradiction, so we are finished.

If \( Q \) is composite, let the prime factorization be: \( Q = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t} \) for primes \( p_i \), \( 1 \leq i \leq t \).

We also note that all odd integers can be written as \( 6k + 1 \), \( 6k + 3 \) or \( 6k + 5 \). But, \( 3 \mid (6k + 3) \), so \( 6k + 3 \) is prime only for \( k = 0 \). So, all primes must be of the form \( 6k + 1 \) or \( 6k + 5 \).

Now, if \( p_1, p_2, \ldots, p_t \) were all of the form \( 6k + 1 \), then \( Q \) would be of the form \( 6k + 1 \), but we know \( Q = 6k + 5 \) for some \( k \), so at least one \( p_i \) is of the form \( 6k + 5 \).
But, \( p_n \times 6 \cdot q_1 q_2 \ldots q_{n-1} \)

\[ \Rightarrow p_n + q_1, q_2, \ldots, q_n \]

\[ \Rightarrow p_n \text{ is a new prime of the form } 6k+5, \]

contradicting the fact that we could write them all in a finite list.

\[ \Rightarrow \exists \text{ infinitely many } 6k+5 \text{ primes.} \]

30. If we write

\[ Q = 4(q_1 q_2 \ldots q_n - 1) + 1 \]

\[ = 4q_1 \ldots q_n + 3 \]

where \( q_i \) are the finitely many primes of the form \( 4k+1 \),

we cannot conclude that \( Q \) is divisible by another prime of this form.

For example, \( 21 = 41k+1 \) for \( k = 5 \)

but \( 21 = 3 \cdot 7 \) and \( 3 \) and \( 7 \) are not of the form \( 4k+1 \).

Other examples will show the proof will not necessarily succeed for certain \( q_1 \ldots q_n \).
Bonus Problems

A. We looked at a solution in class, and I will post an example code on the website.

B. We answered this in class, but the guaranteed minimum is 98. Each mouse counts the number of red hats in front of themselves and keeps track of what is called out. The 99th mouse may or may not die. General solution shown next week.

C. i) 6 people:

![Diagram of 6 people]

ii and iii) After 6 people, we can keep adding groups of 4:

![Diagram of additional groups]

to the party, so the pattern is $6 + 4k$.

We can also have 8 people: so, we can have $8 + 4k$, so any even number $> 8$ is possible.

![Diagram of 8 people]
In general, \((n-1)\) people shake 3 hands, and one who shakes 1 hand

\[
\Rightarrow \frac{3(n-1) + 1}{2} \text{ total handshakes take place}
\]

\[
\Rightarrow \frac{3n}{2} - 1 \text{ handshakes}
\]

\[
\Rightarrow n \text{ must be even}
\]

Therefore, no odd number of people can be there, so 21 is not possible.
D. Cut the bar as follows:

- Cut #1
- Cut #2
- Cut #3
- Cut #4

1 cm  2 cm  4 cm  8 cm  16 cm

Since we can write any number \( \leq 32 \) as 5 binary digits: 00001 \( \Rightarrow \) 11111, we can argue that a 1 cm, 2 cm, 4 cm, 8 cm, and 16 cm piece all suffice.

Hence, from above, only 4 cuts are needed.

E. 3 socks.

F. (i) \( W = 1.5 + \frac{W}{2} \)
   \[ \Rightarrow W = 3.0 \]

(ii) \( W = 1.5 + \frac{W}{3} \)
   \[ \Rightarrow W = 2.25 \]

(iii) \( W = 1.5 + \frac{W}{n} \)
   \[ \Rightarrow \frac{W(n-1)}{n} = 1.5 \]
   \[ \Rightarrow W = \frac{1.5n}{n-1} \]

G. They are arranged alphabetically in English.
   In French: Z, 5, 8, 9, 4, 6, 7, 3, 1, 0

H. She's wearing a firefighter's uniform/hat/...