

Discrete Mathematics

Lecture 2

Logic of Quantified Statements

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Predicates

- A predicate is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables
- The domain of a predicate variable is a set of all values that may be substituted in place of the variable
- $P(x)$: x is a student at NYU

Predicates

- If $P(x)$ is a predicate and x has domain D , the truth set of $P(x)$ is the set of all elements in D that make $P(x)$ true when substituted for x . The truth set is denoted as:
$$\{x \in D \mid P(x)\}$$
- Let $P(x)$ and $Q(x)$ be predicates with the common domain D . $P(x) \Rightarrow Q(x)$ means that every element in the truth set of $P(x)$ is in the truth set of $Q(x)$.
 $P(x) \Leftrightarrow Q(x)$ means that $P(x)$ and $Q(x)$ have identical truth sets

Universal Quantifier

- Let $P(x)$ be a predicate with domain D . A universal statement is a statement in the form “ $\forall x \in D, P(x)$ ”. It is true iff $P(x)$ is true for every x from D . It is false iff $P(x)$ is false for at least one x from D . A value of x from which $P(x)$ is false is called a counterexample to the universal statement
- Examples
 - $D = \{1, 2, 3, 4, 5\}: \forall x \in D, x^2 \geq x$
 - $\forall x \in \mathbb{R}, x^2 \geq x$
- Method of exhaustion

Existential Quantifier

- Let $P(x)$ be a predicate with domain D . An existential statement is a statement in the form “ $\exists x \in D, P(x)$ ”. It is true iff $P(x)$ is true for at least one x from D . It is false iff $P(x)$ is false for every x from D .
- Examples:
 - $\exists m \in \mathbb{Z}, m^2 = m$
 - $E = \{5, 6, 7, 8, 9\}, \exists x \in E, m^2 = m$

Universal Conditional Statement

- Universal conditional statement “ $\forall x$, if $P(x)$ then $Q(x)$ ”:
 - $\forall x \mathbb{R}$, if $x > 2$, then $x^2 > 4$
- Writing Conditional Statements Formally
- Universal conditional statement is called vacuously true or true by default iff $P(x)$ is false for every x in D

Negation of Quantified Statements

- The negation of a universally quantified statement $\forall x \in D, P(x)$ is $\exists x \in D, \sim P(x)$
- “All balls in the bowl are red” – Vacuously True
Example for Universal Statements
- The negation of an existentially quantified statement $\exists x \in D, P(x)$ is $\forall x \in D, \sim P(x)$
- The negation of a universal conditional statement $\forall x \in D, P(x) \rightarrow Q(x)$ is $\exists x \in D, P(x) \wedge \sim Q(x)$

Exercises

- Write negations for each of the following statements:
 - All dinosaurs are extinct
 - No irrational numbers are integers
 - Some exercises have answers
 - All COBOL programs have at least 20 lines
 - The sum of any two even integers is even
 - The square of any even integer is even
- Let $P(x)$ be some predicate defined for all real numbers x , let:
 $r = \forall x \in \mathbb{Z}, P(x)$; $s = \forall x \in \mathbb{Q}, P(x)$; $t = \forall x \in \mathbb{R}, P(x)$
 - Find $P(x)$ (but not $x \in \mathbb{Z}$) so that r is true, but s and t are false
 - Find $P(x)$ so that both r and s are true, but t is false

Variants of Conditionals

- Contrapositive
- Converse
- Inverse
- Generalization of relationships from before
- Examples

Necessary and Sufficient Conditions, Only If

- $\forall x$, $r(x)$ is a sufficient condition for $s(x)$
means: $\forall x$, if $r(x)$ then $s(x)$
- $\forall x$, $r(x)$ is a necessary condition for $s(x)$
means: $\forall x$, if $s(x)$ then $r(x)$
- $\forall x$, $r(x)$ only if $s(x)$ means: $\forall x$, if $r(x)$ then $s(x)$

Multiply Quantified Statements

- For all positive numbers x , there exists number y such that $y < x$
- There exists number x such that for all positive numbers y , $y < x$
- For all people x there exists person y such that x loves y
- There exists person x such that for all people y , x loves y
- Definition of mathematical limit
- Order of quantifiers matters in some (most) cases

Negation of Multiply Quantified Statements

- The negation of $\forall x, \exists y, P(x, y)$
is logically equivalent to $\exists x, \forall y, \sim P(x, y)$
- The negation of $\exists x, \forall y, P(x, y)$
is logically equivalent to $\forall x, \exists y, \sim P(x, y)$

Prolog Programming Language

- Can use parts of logic as programming lang.
- Simple statements:
 `isabove(g, b), color(g, gray)`
- Quantified statements:
 if `isabove(X, Y)` and `isabove(Y, Z)` then
 `isabove(X, Z)`
- Questions:
 `?color(b, blue), ?isabove(X, w)`

Exercises

- Rewrite $\exists!x \in D, P(x)$ without using the symbol $\exists!$
- Determine whether a pair of quantified statements have the same truth values
 - $\forall x \in D, (P(x) \wedge Q(x))$ vs $(\forall x \in D, P(x)) \wedge (\forall x \in D, Q(x))$
 - $\exists x \in D, (P(x) \wedge Q(x))$ vs $(\exists x \in D, P(x)) \wedge (\exists x \in D, Q(x))$
 - $\forall x \in D, (P(x) \vee Q(x))$ vs $(\forall x \in D, P(x)) \vee (\forall x \in D, Q(x))$
 - $\exists x \in D, (P(x) \vee Q(x))$ vs $(\exists x \in D, P(x)) \vee (\exists x \in D, Q(x))$

Arguments with Quantified Statements

- Rule of universal instantiation: if some property is true of everything in the domain, then this property is true for any subset in the domain
- Universal Modus Ponens:
 - Premises: $(\forall x, \text{if } P(x) \text{ then } Q(x)); P(a)$ for some a
 - Conclusion: $Q(a)$
- Universal Modus Tollens:
 - Premises: $(\forall x, \text{if } P(x) \text{ then } Q(x)); \sim Q(a)$ for some a
 - Conclusion: $\sim P(a)$
- Converse and inverse errors

Validity of Arguments using Diagrams

- Premises: All human beings are mortal; Zeus is not mortal. Conclusion: Zeus is not a human being
- Premises: All human beings are mortal; Felix is mortal. Conclusion: Felix is a human being
- Premises: No polynomial functions have horizontal asymptotes; This function has a horizontal asymptote. Conclusion: This function is not a polynomial