Discrete Mathematics
Lecture 1
Logic of Compound Statements

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Logic of Statements

- Logical Form and Logical Equivalence
- Conditional Statements
- Valid and Invalid Arguments
- Digital Logic Circuits
- Number Systems & Circuits for Addition

Logical Form

- Initial terms in logic: sentence, true, false
- Statement (proposition) is a sentence that is true or false but not both
- Compound statement is a statement built out of simple statements using logical operations: negation, conjunction, disjunction

Logical Form

- Truth table
- Precedence of logical operations
- English words to logic:
  - It is not hot ***but*** it is sunny
  - It is ***neither*** hot ***nor*** sunny
- Statement form (propositional form) is an expression made up of statement variables and logical connectives (operators)
- Exclusive OR: XOR

Logical Form

- Truth table for $(\sim p \wedge q) \vee (q \wedge \sim r)$
- Two statements are called logically equivalent if and only if (iff) they have identical truth tables
- Double negation
- Non-equivalence: $\sim(p \vee q)$ vs $\sim p \vee \sim q$
- De Morgan's Laws:
  - The negation of and AND statement is logically equivalent to the OR statement in which component is negated
  - The negation of an OR statement is logically equivalent to the AND statement in which each component is negated

Logical Form

- Applying De-Morgan’s Laws:
  - Write negation for
    - The bus was late or Tom’s watch was slow
    - -1 < x <= 4
- Tautology is a statement that is always true regardless of the truth values of the individual logical variables
- Contradiction is a statement that is always false regardless of the truth values of the individual logical variables

Logical Equivalence

- Commutative laws: $p \wedge q \equiv q \wedge p$, $p \vee q \equiv q \vee p$
- Associative laws: $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$, $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- Distributive laws: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- Identity laws: $p \wedge t \equiv p$, $p \vee c \equiv p$
- Negation laws: $p \vee {\sim}p \equiv t$, $p \wedge {\sim}p \equiv c$
- Double negative law: ${\sim}({\sim}p) \equiv p$
- Idempotent laws: $p \wedge p \equiv p$, $p \vee p \equiv p$
- De Morgan’s laws: ${\sim}(p \wedge q) \equiv {\sim}p \vee {\sim}q$, ${\sim}(p \vee q) \equiv {\sim}p \wedge {\sim}q$
- Universal bound laws: $p \vee t \equiv t$, $p \wedge c \equiv c$
- Absorption laws: $p \vee (p \wedge q) \equiv p$, $p \wedge (p \vee q) \equiv p$
- Negation of t and c: ${\sim}t \equiv c$, ${\sim}c \equiv t$

Conditional Statements

- If something, then something: p → q, p is called the hypothesis and q is called the conclusion
- The only combination of circumstances in which a conditional sentence is false is when the hypothesis is true and the conclusion is false
- A conditional statements is called vacuously true or true by default when its hypothesis is false
- Among $\wedge$, $\vee$, ~ and → operations, → has the lowest priority

Conditional Statements

- Write truth table for: $p \wedge q \rightarrow \sim p$
- Show that $(p \vee q) \rightarrow r = (p \rightarrow r) \wedge (q \rightarrow r)$
- Representation of $\rightarrow$: $p \rightarrow q = \sim p \vee q$
- Re-write using if-else: Either you get in class on time, or you risk missing some material
- Negation of $\rightarrow$: $\sim(p \rightarrow q) = p \wedge \sim q$
- Write negation for: If it is raining, then I cannot go to the beach

Conditional Statements

- Contrapositive p $\rightarrow$ q is another conditional statement ~q $\rightarrow$ ~p
- A conditional statement is equivalent to its contrapositive
- The converse of p $\rightarrow$ q is q $\rightarrow$ p
- The inverse of p $\rightarrow$ q is ~p $\rightarrow$ ~q
- Conditional statement and its converse are not equivalent
- Conditional statement and its inverse are not equivalent

Conditional Statements

- The converse and the inverse of a conditional statement are equivalent to each other
- p only if q means ~q $\rightarrow$ ~p, or p $\rightarrow$ q
- Biconditional of p and q means "p if and only if q" and is denoted as p $\leftrightarrow$ q
- r is a sufficient condition for s means "if r then s"
- r is a necessary condition for s means "if not r then not s"

Exercises

- Write contrapositive, converse and inverse statements for:
  - If P is a square, then P is a rectangle
  - If today is Thanksgiving, then tomorrow is Friday
  - If c is rational, then the decimal expansion of r is repeating
  - If n is prime, then n is odd or n is 2
  - If x is nonnegative, then x is positive or x is 0
  - If Tom is Ann's father, then Jim is her uncle and Sue is her aunt
  - If n is divisible by 6, then n is divisible by 2 and n is divisible by 3

Arguments

- An argument is a sequence of statements. All statements except the final one are called premises (or assumptions or hypotheses). The final statement is called the conclusion.
- An argument is considered valid if from the truth of all premises, the conclusion must also be true.
- The conclusion is said to be inferred or deduced from the truth of the premises

Arguments

- Test to determine the validity of the argument:
  - Identify the premises and conclusion of the argument
  - Construct the truth table for all premises and the conclusion
  - Find critical rows in which all the premises are true
  - If the conclusion is true in all critical rows then the argument is valid, otherwise it is invalid
- Example of valid argument form:
  - Premises: $p \vee (q \vee r)$ and $\sim r$, conclusion: $p \vee q$
- Example of invalid argument form:
  - Premises: $p \rightarrow q \vee \sim r$ and $q \rightarrow p \wedge r$, conclusion: $p \rightarrow r$

Valid Argument-Forms

- Modus ponens (method of affirming):
  - Premises: p → q and p, conclusion: q
- Modus tollens (method of denying):
  - Premises: p → q and ~q, conclusion: ~p
- Disjunctive addition:
  - Premises: p, conclusion: p | q
  - Premises: q, conclusion: p | q
- Conjunctive simplification:
  - Premises: p & q, conclusion: p, q

Valid Argument-Forms

- Disjunctive Syllogism:
  - Premises: p | q and ~q, conclusion: p
  - Premises: p | q and ~p, conclusion: q
- Hypothetical Syllogism
  - Premises: p → q and q → r, conclusion: p → r
- Dilemma: proof by division into cases:
  - Premises: p | q and p → r and q → r, conclusion: r

Complex Deduction

- Premises:
  - If my glasses are on the kitchen table, then I saw them at breakfast
  - I was reading the newspaper in the living room or I was reading the newspaper in the kitchen
  - If I was reading the newspaper in the living room, then my glasses are on the coffee table
  - I did not see my glasses at breakfast
  - If I was reading my book in bed, then my glasses are on the bed table
  - If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table
- Where are the glasses?

Fallacies

- A fallacy is an error in reasoning that results in an invalid argument
- Three common fallacies:
  - Vague or ambiguous premises
  - Begging the question (assuming what is to be proved)
  - Jumping to conclusions without adequate grounds
- Converse Error:
  - Premises: $p \rightarrow q$ and q, conclusion: p
- Inverse Error:
  - Premises: $p \rightarrow q$ and $\sim p$, conclusion: $\sim q$

Fallacies

- It is possible for a valid argument to have false conclusion and for an invalid argument to have a true conclusion:
  - Premises: if John Lennon was a rock star, then John Lennon had red hair, John Lennon was a rock star; Conclusion: John Lennon had red hair
  - Premises: If New York is a big city, then New York has tall buildings, New York has tall buildings; Conclusion: New York is a big city

Contradiction

- Contradiction rule: if one can show that the supposition that a statement p is false leads to a contradiction , then p is true.
- Knight is a person who always says truth, knave is a person who always lies:
  - A says: B is a knight
  - B says: A and I are of opposite types

  What are A and B?

Digital Logic Circuits

- Digital Logic Circuit is a basic electronic component of a digital system
- Values of digital signals are 0 or 1 (bits)
- Black Box is specified by the signal input/output table
- Three gates: NOT-gate, AND-gate, OR-gate
- Combinational circuit is a combination of logical gates
- Combinational circuit always correspond to some boolean expression, such that input/output table of a table and a truth table of the expression are identical

Digital Logic Circuits

- A recognizer is a circuit that outputs 1 for exactly one particular combination of input signals and outputs 0's for all other combinations
- Multiple-input AND and OR gates
- Finding a circuit that corresponds to a given input/output table:
  - Construct equivalent boolean expression using disjunctive normal form: for all outputs of 1 construct a conjunctive form based on the truth table row. All conjunctive forms are united using disjunction
  - Construct a digital logic circuit equivalent to the boolean expression

Digital Logic Circuits

- Design a circuit for the following output: (0, 0, 1, 1, 0, 0, 1, 0)
- Two digital logic circuits are equivalent iff their input/output tables are identical
- Simplification of circuits
- Scheffer stroke (NAND)
- Peirce arrow (NOR)

Number Systems

- Decimal number system
- Binary number system
- Conversion between decimal and binary numbers
- Binary addition and subtraction

Digital Circuits for Addition

- Half Adder – addition of two bits
- Full Adder – addition of two bits and a carry
- Parallel Adder – addition of multi-bit numbers

Negative Numbers

- Two's complement of a positive integer a relative to a fixed bit length n is the binary representation of $2^n - a$
- To find an 8-bit complement:
  - Write 8-bit binary representation of the number
  - Flip all bits (one's complement)
  - Add 1 to the obtained binary
- Addition of negative numbers

Hexadecimal Numbers

- Hexadecimal notation is a number system with base 16
- Digits of hexadecimal number system
- Conversion between hexadecimal and binary and hexadecimal and decimal systems