Homework 9

1. (This problem is adapted from problem 3.4 of Ascher and Petzold).

The implicit midpoint rule satisfies

$$\hat{y}_{n+1} = \hat{y}_n + hf\left(\frac{\hat{y}_n + \hat{y}_{n+1}}{2}\right).$$

(a) Show that the midpoint rule is second-order (you may assume stability and just show consistency) and plot the region of absolute stability in the complex plane. Is the method $A$-stable?

(b) Consider the non-autonomous system

$$y' = \lambda(t)y.$$

Assuming that $\Re(\lambda) \leq 0$, show that for this problem, the midpoint rule satisfies

$$|y_{n+1}| \leq |y_n|.$$

This property is called $AN$-stability. Also show that the trapezoidal rule is not $AN$-stable – that is, for non-autonomous linear systems, the numerical solution can grow even when the true solution decays, even though this cannot happen for autonomous linear systems.

2. In this problem, we will consider the equation for a pendulum,

$$\theta''(t) + \sin(\theta(t)) = 0$$

(a) Consider the energy

$$E = \frac{1}{2}\theta'^2 - \cos(\theta).$$

Show that $\frac{d}{dt}E(\theta(t)) = 0$; that is, for the exact solution, the energy should remain constant over time.

(b) Write programs to solve the ODE using Euler’s method and using the trapezoidal rule. Your program should have the initial condition $\theta_0$ (we will assume $\theta'(0) = 0$), the final time $T$, and the number of steps $N$ as parameters. Note that you will need to
add an auxiliary variable to convert to first-order form. Also, you will need to use a Newton iteration to solve the update formula for the trapezoidal rule. I recommend using an absolute residual tolerance of $10^{-12}$ as the termination criterion for this problem.

(c) Compute the Euler solution and the trapezoidal rule solution to final time $T = 10$ for $N = 200$ and $N = 1000$ with two sets of initial conditions: $\theta_0 = 3.1, \theta_0 = 1$. In each case, you should make a plot of the computed $\theta$ versus $t$ and the computed energy versus $t$. What trends do you notice? How much do you believe the computed solutions?

(d) When $\theta$ is small, we can use the small-angle approximation $\sin(\theta) \approx \theta$, which gives us the approximate solution $\hat{\theta}(t) = \theta_0 \cos(t)$. For small $\theta_0$, this approximation is quite good, and this gives us a way to sanity check our conclusions about the quality of the solution. Rerun the computation for the initial condition $\theta_0 = 10^{-3}$. Make two plots, one for $h = 0.01$ and one for $h = 0.05$, on which you show the solution using Euler’s method, the solution using the trapezoidal rule, and the small-angle approximation $\hat{\theta}(t)$. What do you notice?