Consider the heat equation:
\[ u_t = u_{xx} \quad \text{for } 0 < x < 1, \ 0 < t \leq T \]
\[ u(0, t) = \phi_L(t), \ u(1, t) = \phi_R(t) \]
\[ u(x, 0) = u_0(x). \]

The initial condition is given by
\[ u_0(x) = \sin^2(\pi x) + x \] and the boundary conditions are
\[ \phi_L(t) = 0, \ \phi_R(t) = \frac{1}{2} \left[ 1 + e^{-5t} \right] \]

1. (a) Solve the problem using the Forward Euler method
\[ \frac{U_j^{n+1} - U_j^n}{k} = \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2} + \frac{U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}}{2h^2} \]
for \( j = 1, 2, \ldots, M, \)

Use the parameters \( T = 0.5 \) and \( M = 40. \) Specify the value of \( \lambda \) you used and show a graph of your solution at times \( t = 0, 0.01, 0.04, 0.1 \) and 0.5.

(b) Try solving the problem using \( \lambda = 0.51 \) up to \( T = 0.1. \) Show a graph of your solution at that time.

2. Repeat problem 1 using the Crank-Nicolson method.
\[ \frac{U_j^{n+1} - U_j^n}{k} = \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{2h^2} + \frac{U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}}{2h^2} \]
for \( j = 1, 2, \ldots, M. \)

Demonstrate that your results are second order in both space and time.

3. Consider the two-dimensional heat equation in the domain \( 0 < x < 1, 0 < y < 1: \)
\[ u_t = u_{xx} + u_{yy} \]
\[ u(0, y, t) = 0, \quad u(1, y, t) = 0 \]
\[ u(x, 0, t) = \sin(\pi x), \quad u(x, 1, t) = 0 \]
\[ u(x, y, 0) = (1 - y^2) \sin(\pi x). \]

Choose \( M, \) set \( h_1 = h_2 = h = 1/(M + 1) \) and solve the problem using the Forward Euler method
\[ \frac{U_{j, \ell}^{n+1} - U_{j, \ell}^n}{k} = \frac{U_{j-1, \ell}^n - 2U_{j, \ell}^n + U_{j+1, \ell}^n}{h^2} + \frac{U_{j-1, \ell-1}^n - 2U_{j, \ell-1}^n + U_{j+1, \ell-1}^n}{h^2} \]
for \( j = 1, 2, \ldots, M \) and \( \ell = 1, 2, \ldots, M. \)

Use the parameters \( T = 0.2 \) and \( M = 20. \) Specify the value of \( \kappa \) you used and show a graph of your solution (a surface) at times \( t = 0, 0.01, 0.03, 0.1 \) and 0.2. Also show contour plots of your solution at those times.

4. Write a program for the ADI method applied to this problem. Demonstrate that you are able to use a considerably larger time step than you used in the explicit scheme above. What is the spatial and temporal order of your algorithm? Demonstrate this.