The problems in this homework set are primarily computational. The calculations should be performed by a computer program, not by hand.

You may find it convenient to use the Matlab functions `polyfit` and `polyval`. **Please note that the last coefficient produced by `polyfit` is the constant coefficient, i.e, the coefficients of the polynomial appear in what might be called reverse order.**

**HW5-1.** The purpose of the first two problems is to give you some personal experiences of what can happen with the interpolating polynomial.

Consider the functions $\sin(x)$ and $\cos(x)$, and suppose that we wish to approximate them in the interval $[-1, 2]$ (where the units measure radians). For both $\sin$ and $\cos$, compute the interpolating polynomials of degrees 2, 3, 4, and 5 at equally spaced points in this interval, including the endpoints. List the coefficients of each polynomial and comment on any relationship to the coefficients of the Taylor series expansions. Compute the differences between the exact values of $\sin$ and $\cos$ and their respective interpolating polynomials at 20 equally spaced points in $[-1, 2]$. Do the interpolating polynomials seem to provide a good fit to these functions?

**HW5-2.** Consider the interval $[-1, 1]$ and a (scaled) “Runge’s function”:

$$f = \frac{1}{1 + 25x^2}.$$  \hspace{1cm} (1)

For both (i) and (ii), perform the computations for (a) 3 points, (b) 10 points, and (c) 21 points.

(i) Compute the interpolating polynomial that matches $f$ at the specified number, denoted by $n+1$, of equally spaced points (including the endpoints). Print the interpolation points and the coefficients of this polynomial. Evaluate the polynomial at $2n$ equally spaced points in the interval $[0, 1]$ (NB: half of the original interval), and compute the difference at these points between the interpolating polynomial and the true values of $f$. Please plot the error and comment on your results.

(ii) Compute the interpolating polynomial that matches $f$ at the $n+1$ zeros of the $(n+1)$-st Chebyshev polynomial $T_{n+1}(x)$.

Note: The zeros of $T_{n+1}(x)$ are at the points

$$z_k = \cos\left(\frac{2k + 1}{n + 1} \left(\frac{\pi}{2}\right)\right), \quad k = 0, 1, \ldots, n.$$  

Print the interpolation points and the coefficients of the interpolating polynomial that matches $f$ at these points. Evaluate the polynomial at $2n$ equally spaced points in the interval $[0, 1]$, and compute the difference between the interpolating polynomial and the true values of $f$ at these points. Please plot the error and comment on your results. In addition, please comment on any notable differences between the results of (i) and (ii) in terms of the accuracy of the interpolating polynomial at points other than the interpolation points.
HW5-3. This problem involves calculation of polynomials that approximate a simple function in the interval $[-1, 1]$, using two different norms. Let $f(x) = x^3$, and assume that we are trying to find the “best” quadratic approximation $ax^2 + bx + c$ to $f$ in this interval.

(i) For $x \in [-1, 1]$ and given $a$, $b$, and $c$, define

$$E_{\infty}(f, a, b, c) = \| f(x) - (ax^2 + bx + c) \|_{\infty} = \max_{x \in [-1,1]} |x^3 - (ax^2 + b + c)|.$$

What are the coefficients $a_{\infty}$, $b_{\infty}$, and $c_{\infty}$ of the best quadratic approximation to $f$ in the infinity norm, i.e., for which values of $a$, $b$, and $c$ is $E_{\infty}(f, a, b, c)$ minimized? What is the value of $E_{\infty}(f, a_{\infty}, b_{\infty}, c_{\infty})$? At which points $x$ in $[-1, 1]$ does $|x^3 - (a_{\infty}x^2 + b_{\infty}x + c_{\infty})|$ equal this value?

(ii) For $x \in [-1, 1]$, define

$$E^2_2(f, a, b, c) = \| f(x) - (ax^2 + bx + c) \|_2^2 = \int_{-1}^{1} (x^3 - (ax^2 + b x + c))^2 \, dx.$$

What are the coefficients of the best quadratic approximation to $f$ in the $\ell_2$ norm, i.e., for which values of $a$, $b$, and $c$ is $E^2_2(f, a, b, c)$ minimized? Show how you obtained this answer. What is the value of $E^2_2(f, a, b, c)$ for these coefficients? For a reasonable number of equally spaced points $x$ in $[-1, 1]$, calculate the values of the error $x^3 - (a_{\infty}x^2 + b_{\infty}x + c_{\infty})$ and plot them. What is one noticeable feature of the error?

(iii) What is the value of $E^2_2(f, a_{\infty}, b_{\infty}, c_{\infty})$?

HW5-4. The purpose of this problem is to try a few different quadrature rules.

(i) It is known that

$$\int_1^{b} \frac{dt}{t} = \ln b.$$

For $b = 2$ and $b = 4$, approximate this integral using the midpoint rule, the trapezoidal rule, and Simpson’s rule. In each case, give the computed approximation and the error. Comment on how well the actual errors correspond to the error estimates given on pages 347–348 of Heath (or in class).

(ii) It is also known that

$$\int_0^{\pi} \sin^2 4t \, dt = \frac{\pi}{2}.$$

Explain what would happen if you tried to approximate this integral with any of the rules in part (i). How could you obtain an accurate approximation to this integral?

HW5-5. The purpose of this problem is for you to experience some of the difficulties caused by stiffness in ordinary differential equations.

Let $y(t)$ be a two-dimensional vector

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix},$$
and suppose that you are asked to solve the initial value problem
\[
y' = \begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} = f(y) = \begin{pmatrix} -1001y_1 \\ -y_2 \end{pmatrix}, \quad \text{with} \quad y(0) = \begin{pmatrix} 1.1 \\ 10.1 \end{pmatrix}.
\]
Assume that you are using a constant step size \( h \), and that you wish to obtain an approximation to \( y \) at the points \( t_0 = 0, t_1 = h, t_2 = 2h \), and so on.

(i) What is the exact solution \( y(t) \)?

(ii) In this part, use the forward (explicit) Euler method,
\[
y_{k+1} = y_k + hf(t_k, y_k)
\]
to obtain the approximate solution.

(a) Give a formula for the forward Euler step from \( y_k \).

(b) Compute the approximate solution for \( h = 10^{-2}, h = 10^{-4}, \) and \( h = 10^{-6} \), for at least 20 steps, starting at \( t = 0 \). Print \( t_k, y_k, y(t_k) \), and the error \( y(t_k) - y_k \) for each point. Please comment on the accuracy of your results, and on what is happening to the error over time.

(iii) Perform the same calculations as in (ii), but using the backward (implicit) Euler method,
\[
y_{k+1} = y_k + hf(t_{k+1}, y_{k+1}).
\]

(a) Give a formula for the implicit Euler step from \( y_k \).

(b) Compute the approximate solution for \( h = 10^{-2}, h = 10^{-4}, \) and \( h = 10^{-6} \), for at least 20 steps. Print \( t_k, y_k, y(t_k) \), and the error \( y(t_k) - y_k \) for each point. Please comment on the accuracy of your results, and on what is happening to the error over time.

(iv) Please comment on the difference in the behavior of the errors for a given \( h \) for the explicit and implicit Euler methods.