Theory of Computation
Homework 10.

Due Date: Monday, November 10.

Chapter 3:
No. 21 part i.
No. 23.
No. 25a part iv.
No. 25b part iii.

Challenge problem: This question seeks to show that DPDAs are closed under complements (the “proof” in the text is incomplete).

The following notation will be helpful: let $\Sigma^+$ denote $\Sigma \cup \{\lambda\}$ and $\Gamma^+$ denote $\Gamma \cup \{\lambda\}$.

Also, for short, we write edge labels “Read $a$, Pop $B$, Push $C$” as $(a, B, C)$.

i. Given a DPDA $M$ we create a new DPDA $M'$ which recognizes the same language but in which there are twice as many nodes, categorized by whether their out-edges have a Read $\lambda$ label, or a Read $a$ label, for some $a \in \Sigma$. Specifically, for each vertex $u$ in $M$, we create vertices $u\lambda$ and $u\neg\lambda$. Each edge $(u, v)$ in $M$ labeled $(a, B, C)$, with $a \in \Sigma$ and $B, C \in \Gamma^+$, is replaced by edges $(u\lambda, v\neg\lambda)$ and $(u\neg\lambda, v\lambda)$ labeled $(a, B, C)$, while if $a = \lambda$ the new edges are $(u\lambda, v\lambda)$ and $(u\neg\lambda, v\lambda)$, again labeled $(a, B, C)$.

Choose a suitable start vertex and suitable recognizing vertices and then show that for the resulting $M'$, $L(M') = L(M)$.

The next few steps simplify the graph for $M'$ while leaving the language it recognizes unchanged.

ii. Show that if edge $(v, w\lambda)$ has label $(\lambda, \lambda, C)$, with $C \in \Gamma^+$, then $v$ has no other out-edge.

iii. Show that if edge $(v, w\lambda)$ has label $(\lambda, \lambda, \lambda)$, then by replacing all edges $(u, v)$ by suitably labeled edges $(u, w)$, node $v$ can be removed from $M'$ while leaving the recognized language unchanged.

iv. Similarly, show that if there is a cycle $(v_1^\lambda, v_2^\lambda), \ldots, (v_i^\lambda, v_{i+1}^\lambda), \ldots, (v_k^\lambda, v_1^\lambda)$ with labels $(\lambda, \lambda, A_i)$, $A_i \in \Gamma$, then all the edges on the cycle can be replaced by new edges to a sink vertex, with all the new edges having label $(\lambda, \lambda, \lambda)$, while leaving the recognized language unchanged.

v. Suppose edge $(v, w\lambda)$ has label $(\lambda, B, C)$, with $B \in \Gamma^+$, $C \in \Gamma$. Suppose further there is an edge $(w\lambda, x\lambda)$ with label $(\lambda, C, D)$ with $D \in \Gamma^+$. Then show that edge $(v, w\lambda)$ with label $(\lambda, B, C)$ can be replaced by edge $(v, x\lambda)$ with label $(\lambda, B, D)$ while leaving the recognized language unchanged. Also note that if $B = D = \lambda$ then, by (iii), vertex $v$ can be removed.

vi. Deduce that in any computation path, if the edge $(w\lambda, x\lambda)$ follows $(v, w\lambda)$, where $(v, w\lambda)$ has label $(\lambda, B, C)$, with $B \in \Gamma^+$ and $C \in \Gamma$, then $(w\lambda, x\lambda)$ has label $(\lambda, \lambda, D)$ for some $D \in \Gamma$. Further deduce, from (i) and (iv), that the computation path must reach an edge with label $(a, E, F)$ with $a \in \Sigma$ and $E, F \in \Gamma^+$.

vii. Conclude that all computations in the modified $M'$ have finite length and hence that swapping recognizing and non-recognizing vertices causes the resulting machine $M'$ to recognize $L$. 