7. Let $G-UHP$ be the following problem.

**Input:** $G = (V, E)$, an undirected graph.

**Question:** Does $G$ have vertices $u$ and $v$ such that there is a Hamiltonian Path from $u$ to $v$?

Suppose that you were given a polynomial time algorithm for Undirected Hamiltonian Path (UHP), the variant of the problem in which the start and end vertices are specified. Using it as a subroutine, give a polynomial time algorithm for $G-UHP$.

**Sample solution, Problem 1, Chapter 5.** Let Half-SAT be the following language:

$$\text{Half-SAT} = \{ F \mid F \text{ is a CNF formula with } 2n \text{ variables and there is a satisfying assignment in which } n \text{ variables are set to True and } n \text{ variables are set to False} \}.$$  

Show that Half-SAT is NP-Complete, that is (i) show that Half-SAT has a polynomial time verifier, and (ii) Supposing that you were given a polynomial time algorithm for Half-SAT, use it to give a polynomial algorithm for SAT.

**Sample Solution.**

i. The certificate comprises an assignment $\sigma$ of truth values to the variables in $F$. Since $|\sigma| = O(|F|)$, the certificate has length linear in $|F|$.

To check that a candidate certificate is correct, the verifier checks (a) that $F$ is a CNF formula with an even number of variables; (b) that the certificate has exactly $n$ variables set to True and $n$ set to False; (c) that the assignment of truth values in the certificate causes $F$ to evaluate to True. It is straightforward to implement the verifier to run in polynomial time, and clearly it accepts exactly when $F \in \text{Half-SAT}$.

ii. Let $F'$ be the input to SAT. The algorithm $A_{\text{sat}}$ to test if $F' \in \text{SAT}$ builds a CNF formula $F$ with the property that

$$F' \in \text{SAT} \iff F \in \text{Half-SAT}.$$  

It then runs the algorithm for Half-SAT, $A_{\text{h-sat}}$, on input $F$ and reports the answer as its own result.

$F$ is built as follows. Suppose that $F'$ has variables $x_1, x_2, \ldots, x_n$. $F$ will have variables $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$. The idea is that $x_i = \overline{y_i}$ for all $i$, $1 \leq i \leq n$. This is enforced by including clauses $(x_i \lor y_i) \land (\overline{x_i} \lor \overline{y_i})$ for $1 \leq i \leq n$. For $(x_i \lor y_i) \land (\overline{x_i} \lor \overline{y_i})$ evaluates to True only if $x_i = \overline{y_i}$ and $\overline{y_i} = \overline{x_i}$ or if $x_i = \overline{y_i}$ and $\overline{y_i} = \overline{x_i}$, i.e. only if $x_i = \overline{y_i}$.

In addition, for each clause $(l_a \lor l_b \lor l_c)$ in $F$, where $l_i$ is one of $x_i$ or $\overline{x_i}$, there will be clauses $(l_a \lor l_b \lor l_c) = (l_{xa} \lor l_{xb} \lor l_{xc})$ and $(\overline{l_ya} \lor \overline{l_yb} \lor \overline{l_yc})$, where $l_{ya}$ denotes $y_i$ if $l_{xi}$ denotes $x_i$ and $l_{yi}$ denotes $\overline{y_i}$ if $l_{xi}$ denotes $\overline{x_i}$.

Clearly $F$ can be built in polynomial time. Also, clearly, if $A_{\text{h-sat}}$ runs in polynomial time, then so does $A_{\text{sat}}$.

Now we argue that $F' \in \text{SAT} \iff F \in \text{Half-SAT}$. 

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It is readily seen that if $F'$ has a satisfying assignment $\sigma$, setting truth values for the $x_i$ in $F$ according to $\sigma$, and then for the $y_i$ according to the rule $y_i = \overline{x_i}$ produces a satisfying assignment for $F$ with exactly $n$ variables set to True. Thus $F' \in \text{SAT}$ implies $F \in \text{Half-SAT}$.

Likewise, a satisfying assignment for $F$ restricted to the $x$ variables is a satisfying assignment for $F'$. Thus $F \in \text{Half-SAT}$ implies $F' \in \text{SAT}$.

This shows that $F' \in \text{SAT} \iff F \in \text{Half-SAT}$. 