Review

- Number Theory
- Natural Numbers with Successor
- Natural Numbers with Successor and Less-Than
- Presburger Arithmetic
Outline

- A Subtheory of Number Theory
- Representable Relations
- Church’s Thesis Revisited
- Representable Functions
- A Catalog of Representable Sets

Source: Enderton, 3.3.
A Subtheory of Number Theory

Let us now add $\times$ and $E$ to our set of symbols. We now have the full language of number theory.

The intended model is $\mathbb{N} = (\mathbb{N}; 0, S, <, +, \times, E)$.

We could do with fewer symbols. For example, $0, S, <$, and $E$ are all definable in $(\mathbb{N}; +, \times)$. However, having all these parameters will simplify some of the proofs.

As we will see, $\text{Th} \mathbb{N}$ is neither decidable nor axiomatizable. This is not at all obvious and will require new and clever techniques to show.

We begin with a finitely axiomatizable subtheory.
A Subtheory of Number Theory

Consider the following set $A_E$ of axioms.

- **S1.** $\forall x \mathbf{S}x \neq \mathbf{0}$.
- **S2.** $\forall x \forall y (\mathbf{S}x = \mathbf{S}y \rightarrow x = y)$.
- **L1.** $\forall x \forall y (x < \mathbf{S}y \leftrightarrow x \leq y)$
- **L2.** $\forall x x \not< 0$
- **L3.** $\forall x \forall y (x < y \lor x = y \lor y < x)$
- **A1.** $\forall x x + \mathbf{0} = x$
- **A2.** $\forall x \forall y x + \mathbf{S}y = \mathbf{S}(x + y)$
- **M1.** $\forall x x \times \mathbf{0} = \mathbf{0}$
- **M2.** $\forall x \forall y x \times \mathbf{S}y = x \times y + x$
- **E1.** $\forall x x E \mathbf{0} = \mathbf{S}0$
- **E2.** $\forall x \forall y x E(\mathbf{S}y) = (xEy) \times x$
A Subtheory of Number Theory

Lemma

1. $A_E \vdash \forall x \ x \neq 0$.

2. $A_E \vdash \forall x \ (x < S^{k+1}0 \leftrightarrow x = S^00 \vee \ldots \vee x = S^k0)$.

Proof

(1) is from L2. For (2), we use regular mathematical induction. For the base case, we must show

$$x < S0 \leftrightarrow x = 0$$

which follows from L1 and L2.

For the inductive step, apply L1 to get

$$x < S^{k+1}0 \leftrightarrow x < S^k0 \vee x = S^k0.$$

The result then follows from the inductive hypothesis. \[\square\]

Notice that we are implicitly using the fact that if $\Gamma \vdash (\phi \leftrightarrow \psi)$, then we can substitute $\psi$ for $\phi$. Why is this true?
Lemma

For any variable-free term $t$, there is a unique natural number $n$ such that $A_E \vdash t = S^n0$.

Proof

Uniqueness is by S1 and S2. Existence is a simple induction argument on $t$.  

Theorem

For any quantifier-free sentence $\tau$ true in $N$, $A_E \vdash \tau$.

Proof

Homework!
A Subtheory of Number Theory

To simplify things, we introduce the following notation for substitution:

\[
\phi(t) = \phi^v_1, \\
\phi(t_1, t_2) = \phi^v_{t_1}^v_{t_2},
\]

etc.

In cases where \(x\) is not substitutable for \(v_1\) in \(\phi\), we assume \(\phi(x) = \psi^v_x\) where \(\psi\) is a suitable alphabetic variant of \(\phi\).

Also, we will make use of the following fact:

For a formula \(\phi\) in which at most \(v_1, \ldots, v_n\) occur free and for natural numbers \(a_1, \ldots, a_n\),

\[
\models_N \phi[[a_1, \ldots, a_n]] \iff \models_N \phi(S^{a_1}0, \ldots, S^{a_n}0).
\]
A Subtheory of Number Theory

We can now extend our results about what formulas are deducible in $A_E$. An existential formula is one of the form $\exists x_1 \cdots \exists x_k \theta$ where $\theta$ is quantifier-free.

**Corollary**

If $\tau$ is an existential sentence true in $N$, then $A_E \vdash \tau$.

**Proof**

If $\exists v_1 \exists v_2 \theta$ is true in $N$, then for some $m, n \in N$, $\theta(S^m0, S^n0)$ is true in $N$. By the previous theorem, this formula is deducible from $A_E$. But it in turn derives $\exists v_1 \exists v_2 \theta$, so $A_E \vdash \exists v_1 \exists v_2 \theta$. □

A universal formula is one of the form $\forall x_1 \cdots \forall x_k \theta$ where $\theta$ is quantifier-free. There are some universal sentences that are true in $N$, but are not deducible from $A_E$. 
Representable Relations

Let $R$ be an $m$-ary relation on $\mathbb{N}$.

$R$ is definable in $\mathbb{N}$ iff there exists some formula $\phi$ in which only $v_1, \ldots, v_n$ occur free such that

$$\langle a_1, \ldots, a_m \rangle \in R \iff \models \mathbb{N} \phi[[a_1, \ldots, a_m]].$$

Notice that this is equivalent to $\models \mathbb{N} \phi(S^{a_1}0, \ldots, S^{a_m}0)$.

So, if $\phi$ defines $R$, it follows that

$$\langle a_1, \ldots, a_m \rangle \in R \Rightarrow \models \mathbb{N} \phi(S^{a_1}0, \ldots, S^{a_m}0),$$

and

$$\langle a_1, \ldots, a_m \rangle \notin R \Rightarrow \models \mathbb{N} \neg \phi(S^{a_1}0, \ldots, S^{a_m}0).$$

*Representability* is the dual notion of *definability* in which truth in a model is replaced by deducibility in a theory.

We say that $\phi$ represents $R$ in a theory $T$ (in a language including $0$ and $S$) iff for every $a_1, \ldots, a_m \in \mathbb{N}$:

$$\langle a_1, \ldots, a_m \rangle \in R \Rightarrow \phi(S^{a_1}0, \ldots, S^{a_m}0) \in T,$$

and

$$\langle a_1, \ldots, a_m \rangle \notin R \Rightarrow (\neg \phi(S^{a_1}0, \ldots, S^{a_m}0)) \in T.$$

A relation is *representable* in $T$ iff there exists some formula that represents it in $T$. 
Representable Relations

Example

The equality relation on $\mathcal{N}$ is represented in $\text{Cn } A_E$ by the formula $v_1 = v_2$.

Clearly, if $m = n$, then $S^m 0 = S^n 0$ is derivable. On the other hand, if $m \neq n$, then from S1 and S2, we can derive $\neg S^m 0 = S^n 0$. 
Representable Relations

A formula $\phi$ in which at most $v_1, \ldots, v_m$ occur free is *numeralwise determined* by $A_E$ iff for every $m$-tuple $a_1, \ldots, a_m$ of natural numbers, either

$$A_E \vdash \phi(S^{a_1}0, \ldots, S^{a_m}0), \text{ or } A_E \vdash \neg \phi(S^{a_1}0, \ldots, S^{a_m}0).$$

**Theorem**

A formula $\phi$ represents a relation $R$ in $CnA_E$ iff

1. $\phi$ is numeralwise determined by $A_E$, and

2. $\phi$ defines $R$ in $N$.

**Proof**

Suppose $\phi$ represents $R$ in $CnA_E$. (1) holds by definition. (2) holds by soundness and the fact that $N$ is a model of $A_E$. On the other hand, given (1) and (2), we have

$$\langle a_1, \ldots, a_m \rangle \in R \Rightarrow \models_N \phi(S^{a_1}0, \ldots, S^{a_m}0) \text{ by (2)}$$

$$\Rightarrow A_E \not\vdash \neg \phi(S^{a_1}0, \ldots, S^{a_m}0) \text{ since } N \models A_E$$

$$\Rightarrow A_E \vdash \phi(S^{a_1}0, \ldots, S^{a_m}0) \text{ by (1)}$$

The case for $\langle a_1, \ldots, a_m \rangle \notin R$ is similar. \qed
Church’s Thesis Revisited

Representability can be linked to decidability.

**Theorem**

If $R$ is representable in a consistent axiomatizable theory, then $R$ is decidable.

**Proof**

Suppose $R$ is represented by $\phi$ in a consistent axiomatizable theory $T$. We know that $T$ is effectively enumerable. Thus, a decision procedure is to enumerate $T$ until either $\phi(S^{a_1}0, \ldots, S^{a_m}0)$ or $\neg\phi(S^{a_1}0, \ldots, S^{a_m}0)$ appears. Since one of these has to be in $T$ by definition of representability, the procedure will eventually terminate. 

□
Church’s Thesis Revisited

We now define the fundamental notion of recursive relations.

A relation $R$ on the natural numbers is recursive iff it is representable in some consistent finitely axiomatizable theory (in a language with 0 and $S$).

Earlier, we gave Church’s thesis as the assertion that all models of computation are either equivalent to or less powerful than what can be done on a computer with infinite time and memory. We can now be more precise.

**Church’s thesis**

A relation is decidable iff it is recursive.

The definition of recursive relations is one of many equivalent formalizations of the notion of decidability.

Our goal is to show that recursiveness is equivalent to representability in $Cn A_E$. This is no small task and will require a significant amount of work. We begin with some basic facts about representability.