1. **Papers and reviewers.** You are the program chair of SODA, the leading algorithms conference! Part of your job is to assign papers to reviewers. You have papers $P_1, \ldots, P_n$ and reviewers $R_1, \ldots, R_k$. Initially, each reviewer constructs a list of papers he is willing to review. An assignment of papers to reviewers is valid if each paper is assigned to at least 3 distinct reviewers that are willing to review that paper. Of course, a valid assignment exists iff each paper has at least 3 willing reviewers — this is easy to check, and we assume that this is the case (if not, as program chair, you have to persuade your reviewers to add more papers to their lists).

Among valid assignments of papers to reviewers, some distribute the work more fairly than others. Let us define the **workload** of an assignment to be the maximum number of papers assigned to any one reviewer (i.e., if $w_i$ papers are assigned to reviewer $R_i$, the workload of the assignment is $\max\{w_i : i = 1, \ldots, k\}$).

Design an efficient algorithm to find a valid assignment with minimum workload. Your algorithm should run in time $O((n + k)^c)$ for some constant $c$.

2. Let

$$\text{SetPACK} = \{\langle C, k \rangle : C \text{ is a collection of finite sets, at least } k \text{ of which are pairwise disjoint}\}.$$ 

Show that \text{SetPACK} is NP-complete.

3. CLRS: Problem 34.4, parts (a) and (b). (Note that you already worked parts (c) and (d) in CLRS Problem 15.7.)


5. Sipser: Problem 7.27.


7. For positive integer $k$, let \text{$k$RSAT} be the set of encodings $\langle \phi \rangle$, such that $\phi$ is a satisfiable Boolean formula in conjunctive normal form, where each variable occurs in at most $k$ distinct clauses. Show that \text{3RSAT} is NP-complete.

8. **Resolution.** This problem develops the foundations underlying a basic technique, called resolution, that is used in automated theorem proving, along with some algorithmic applications.

   (a) Let $A_1, \ldots, A_m$, $B_1, \ldots, B_n$, and $C$ be arbitrary Boolean formulas, none of which contain the variable $x$. Show that the formula

   $$(\bigwedge_{i=1}^{m} (x \lor A_i)) \land \left(\bigwedge_{j=1}^{n} (\overline{x} \lor B_j)\right) \land C$$

   is satisfiable if and only if the formula

   $$\left(\bigwedge_{1 \leq i \leq m \atop 1 \leq j \leq n} (A_i \lor B_j)\right) \land C$$

   is satisfiable.

   (b) Using part (a), show that \text{2RSAT} (see previous exercise) is in P.

   (c) Let \text{2SAT} be the set of encodings $\langle \phi \rangle$, such that $\phi$ is a satisfiable Boolean formula in conjunctive normal form, where each clause consists of two distinct literals. Using part (a), show that \text{2SAT} in P.

9. Recall that a Boolean formula $\phi$ is 3CNF formula if it is in conjunctive normal form, and each clause contains three distinct literals. Suppose $\phi$ is a 3CNF formula with $k$ clauses. Let $\alpha$ be a nonnegative number. We say that $\phi$ is $\alpha$-satisfiable if there is an assignment that satisfies at least $\alpha k$ clauses of $\phi$.

   (a) Let $\phi$ be a 3CNF formula with $k$ clauses and $m$ variables. Suppose we choose a random assignment to the variables (chosen uniformly from among all $2^m$ assignments), and let $T$ be the random variable representing the number of clauses of $\phi$ that are satisfied by this assignment. Consider the expected value $E[T]$ of $T$. Show that $E[T] \geq (7/8)k$. 


(b) Using the result of part (a), argue that $\phi$ is $7/8$-satisfiable.

(c) Using the result of part (a), give an efficient randomized algorithm that on input $\phi$, actually finds an assignment that satisfies at least $(7/8)k$ clauses of $\phi$.

(d) Same as part (c), but your algorithm should be deterministic (but still polynomial time).

10. Let $8/9$-SAT be the set of encodings $\langle \phi \rangle$, such that $\phi$ is an $8/9$-satisfiable 3CNF formula (see previous exercise). Show that $8/9$-SAT is NP-complete.