All pairs shortest paths

One approach:

- Run a single-source shortest path algorithm from each vertex
  - Bellman-Ford (no negative cycles): $O(|V|^2(|V| + |E|))$, or $O(|V|^4)$
  - Dijkstra (no negative edges): $O(|V|(|V| \log |V| + |E|))$, or $O(|V|^3)$

Floyd-Warshall Algorithm:

- no negative cycles
- running time $O(|V|^3)$
• Number the vertices 1 .. n
• For a path \( \rho = (v_0, v_1, \ldots, v_{k-1}, v_k) \), we say that \( v_1, \ldots, v_{k-1} \) are intermediate vertices
• For \( k = 0 \ldots n \), \( \delta^{(k)}(i, j) := \) length of the shortest path from \( i \) to \( j \) whose intermediate vertices belong to \( \{1, \ldots, k\} \)

\[
\delta^{(0)}(i, j) = \begin{cases} 
0 & \text{if } i = j; \\
w(i, j) & \text{if } i \neq j \text{ and } (i, j) \in E \\
\infty & \text{otherwise}
\end{cases}
\]

• For \( k > 0 \)
\[
\delta^{(k)}(i, j) = \min \left( \begin{array}{c}
\delta^{(k-1)}(i, j), \\
\delta^{(k-1)}(i, k) + \delta^{(k-1)}(k, j)
\end{array} \right)
\]
Straightforward implementation:

- Use a 3D array $D[i, j, k]$

$$D[i, j, 0] \leftarrow \delta^{(0)}(i, j) \text{ for } i, j = 1 \ldots n$$

for $k \leftarrow 1 \text{ to } n$ do

- for $i \leftarrow 1 \text{ to } n$ do

  - for $j \leftarrow 1 \text{ to } n$ do

    - for $j' \leftarrow 1 \text{ to } n$ do

      $d' \leftarrow D[i, k, k - 1] + D[k, j, k - 1]$

      if $d' < D[i, j, k - 1]$

      then $D[i, j, k] \leftarrow d'$

      else $D[i, j, k] \leftarrow D[i, j, k - 1]$

- Running time: $O(n^3)$

- Space: $O(n^3)$
Improving the space requirement:

- Since $D[\cdot, \cdot, k]$ depends only on $D[\cdot, \cdot, k-1]$, we can obviously get by with just two 2D arrays.
- In fact, we can get by with just a single array, with updates “in place”.

Justification:

- $\delta^{(k)}(i, i) = 0$ for all $i, k$ \textbf{(no negative cycles)}
- $\delta^{(k)}(i, k) = \min(\delta^{(k-1)}(i, k), \delta^{(k-1)}(i, k) + \delta^{(k-1)}(k, k)) = \delta^{(k-1)}(i, k)$
- $\delta^{(k)}(k, j) = \delta^{(k-1)}(k, j)$
Improved implementation:

- Use a 2D array $D[i, j]$

\[
D[i, j] \leftarrow \delta^{(0)}(i, j) \text{ for } i, j = 1 \ldots n
\]

for $k \leftarrow 1$ to $n$ do

for $i \leftarrow 1$ to $n$ do

for $j \leftarrow 1$ to $n$ do

\[
d' \leftarrow D[i, k] + D[k, j]
\]

if $d' < D[i, j]$

then

\[
D[i, j] \leftarrow d'
\]
Adding path recovery:

- Two arrays: $D[i, j], N[i, j]$
  
  $D[i, j] \leftarrow \delta^{(0)}(i, j)$ for $i, j = 1 \ldots n$

  $N[i, j] \leftarrow j$ for $i, j = 1 \ldots n$

  for $k \leftarrow 1$ to $n$ do
    for $i \leftarrow 1$ to $n$ do
      for $j \leftarrow 1$ to $n$ do
        $d' \leftarrow D[i, k] + D[k, j]$
        if $d' < D[i, j]$ then
          $D[i, j] \leftarrow d'$
          $N[i, j] \leftarrow N[i, k]$

Printing a shortest path from $u$ to $v$:

$x \leftarrow u$, print $x$

while $x \neq v$ do: $x \leftarrow N[x, v]$, print $x$
Johnson’s Algorithm

Motivation:
- For sparse graphs with no negative edges, running Dijkstra from each node is faster than Floyd-Warshall
- But what if we have a sparse graph with negative edges, and no negative cycles?

Idea:
- Convert $G$ to a graph $G'$ such that
  - $G'$ has no negative edges
  - Shortest paths in $G'$ are shortest paths in $G$
- How? Re-weighting
• Let $G = (V, E)$ be a directed graph with weights $w : E \rightarrow \mathbb{R}$, and no negative cycles

• Let $h : V \rightarrow \mathbb{R}$, and define
  
  $\hat{w}(u, v) = w(u, v) + h(u) - h(v)$

• For a path $p$ in $G$, define $\hat{w}(p)$ to be the weight of the path, using $\hat{w}$

• Observation: if $p$ is a path from $u$ to $v$, then
  
  $\hat{w}(p) = w(p) + h(u) - h(v)$ (telescoping sum)

• No negative cycles relative to $\hat{w}$

• Define $\hat{\delta}(u, v)$ to be the length of the shortest path from $u$ to $v$, relative to $\hat{w}$

• Therefore, any shortest path relative to $\hat{w}$ is also a shortest path relative to $w$, and
  
  $\hat{\delta}(u, v) = \delta(u, v) + h(u) - h(v)$
Johnson’s Algorithm:

- Construct a new weighted graph $G'$ by adding a node $s$ to $G$ and 0-weight edges connecting $s$ to every other vertex in $G$
- Run Bellman-Ford starting from $s$, to obtain $\delta(s, \nu)$ for all nodes $\nu$
- Define $h(\nu) := \delta(s, \nu)$, and re-weight as above
- By Triangle Inequality, for every edge $(u, \nu)$:
  $\delta(s, \nu) \leq \delta(s, u) + w(u, \nu) = h(u) + w(u, \nu)$
  and so
  \[ \hat{w}(u, \nu) = w(u, \nu) + h(u) - h(\nu) \geq 0 \]
- Run Dijkstra from every node, using $\hat{w}$
Running time:

- Bellman-Ford: $O(|V|(|V| + |E|))$
- Dijkstra: $O(|V|(|V| \log |V| + |E|))$
- Dijkstra dominates
Problem

• You are given a directed graph $G = (V, E)$ with non-negative weights $w : E \to \mathbb{R}_{\geq 0}$, along with nodes $s, t \in V$, and a positive integer $k$

• Edges are colored red and green

• A path is called admissible if it contains at most $k$ red edges

• Determine if there is an admissible path from $s$ to $t$, and if so, find one that is shortest (relative to $w$)