1. This is another exercise on induction.
   Consider the Floyd-Warshall algorithm for the all pairs shortest path problem. It com-
putes the length of a shortest path from vertex \( v_i \) to vertex \( v_j \) for every pair \((v_1, v_j)\) of vertices
in an \( n \)-vertex directed graph \( G \), assuming there are no negative length cycles in the graph.
The length of the edge from \( v_i \) to \( v_j \), if any, is the \((i, j)\)th entry in an \( n \times n \) array \( L \); if there
is no such edge, the corresponding entry in array \( L \) has value \( \infty \).
Recall that the algorithm, in its \( k \)th iteration \((1 \leq k \leq n)\) computes the length of a short-
ext path that is limited to intermediate vertices drawn from the vertex set \( \{v_1, v_2, \ldots, v_k\} \). Let \( S_k(i, j) \) denote the length of a shortest path from \( v_i \) to \( v_j \) as computed in the \( k \)th iteration.
Then the algorithm can be written as follows.

initialize:
\[
\text{for } 1 \leq i \leq n, 1 \leq j \leq n \text{ do}
\quad S_0(i, j) \leftarrow L(i, j)
\text{end}
\]

for \( k = 1 \) to \( n \) do
\[
\text{for } 1 \leq i \leq n, 1 \leq j \leq n \text{ do}
\quad S_k(i, j) \leftarrow \min\{S_{k-1}(i, j), S_{k-1}(i, k) + S_{k-1}(k, j)\}
\text{end}
\]
\[
\text{end}
\]

Prove by induction that at the end of the computation \( S_n(i, j) \) stores the length of a shortest
path from \( v_1 \) to \( v_j \) for all pairs of vertices \((v_i, v_j), 1 \leq i, j \leq n\).
Your proof needs to take the following form.
It needs to begin with a statement of the inductive hypothesis. This involves an inductive
index.
Next comes a proof of a suitable base case, for an appropriate small value of the inductive
index (in general, one may need to prove a base case for several small index values).
This is followed with a proof for the inductive step. Begin this with a statement of what
is being assumed (i.e. for what values of the inductive index the inductive hypothesis is
assumed to be true). Follow this with a statement of what you are going to prove, namely
that the inductive hypothesis is going to be shown true for a larger index value. Don’t just
repeat the previous sentence; you need to spell out what is the inductive hypothesis in this
case. At this point you can prove your claim.
Finally, conclude that the result is proven, stating what it is.
I intend to grade this problem by commenting on your solution. If it could be improved,
I will ask you to rewrite it and resubmit it. I will continue iterating this process until your
solution is sufficiently good. Note that coherent English is a part of any good solution.
Lastly, you may and should use my previous solutions as example templates.

4. Let $A = \{a^i w \mid w \in \{a,b\}^* \text{ and } w \text{ contains at most } i \text{ a’s}, i \geq 1\}$. Show that $A$ is not regular. Remember that if $B$ is regular and if $A$ is regular then so is $A \cap B$. It may be helpful to choose a suitable $B$, and then show that $A \cap B$ is not regular.