Honors Algorithms — G22.3520-010 Fall 2006 — Problem Set 7
Due: Dec. 13

1. **Divisibility by n.** For $x \in \{0,1\}^*$, let $I(x)$ be the integer value of x, interpreted as a binary number (with $I(\epsilon) := 0$). Observe that $I(xy) = 2^k I(x) + I(y)$, where $k := |y|$. For integer $n \geq 1$, let

$$B_n := \big\{x \in \{0,1\}^* : I(x) \equiv 0 \pmod{n}\big\}.$$

   (a) Design a DFA that recognizes B_n. Your DFA should have (at most) n states.

   (b) Show that if $n = 2^k$, there is a DFA with $k+1$ states that recognizes B_n.

   (c) Show that if n is odd, there is no DFA with fewer than n states that recognizes B_n.

2. **A nonregular language.** Let $I(x)$ be defined as in the previous exercise. Show that $S := \{x : I(x) = m^2 \text{ for some } m \in \mathbb{Z}\}$ is not regular.

3. **A funny closure property.** If A is a language, let

$$A_{1/2} := \{x : xy \in A \text{ for some } y \text{ with } |y| = |x|\}.$$

   Show that if A is regular, then so is $A_{1/2}$.

4. **Edit distance to a regular language.** You are given a NFA M, represented as a directed graph with vertices Q and edges E, along with a start state $q_0 \in Q$ and a set $F \subset Q$ of accept states. Assume that each edge has one label: either a symbol of the underlying (fixed) alphabet Σ or the empty string ϵ. You are also given a string $x \in \Sigma^*$. The task is to compute a string $y \in L(M)$ of minimum edit distance from x (or determine that no such y exists, which happens only if $L(M) = \emptyset$). Here, the *edit distance* between x and y is the minimum number of single-symbol insertions, deletions, or changes needed to transform x into y. Show how to solve this problem in time $O((|x|+1)(|Q|+|E|))$.

5. **Some contex-free languages** For $w \in \{0,1\}^*$, let $N_0(w)$ (resp., $N_1(w)$) be the number of times 0 (resp., 1) appears in w.

   (a) Let $A = \{w \in \{0,1\}^* : N_0(w) = 2N_1(w)\}$. Give a CFG for A, and a careful proof that your grammar is correct.

   (b) Let $B = \{w \in \{0,1\}^* : 0.4N_1(w) \leq N_0(w) \leq N_1(w)\}$. Give a PDA for B.

   (c) Let $C = \{x\#y : x, y \in \{0,1\}^*, x \neq y\}$. Show that C is context free.

6. **Some languages that are not context free.**

   (a) Let A be the language of all strings over $\{a,b,c,d\}$ such that the number of a's equals the number of b's, and the number of c's equals the number of d's. Show that A is not context free.

   (b) Let $B = \{0^{nm}1^m : n, m \in \mathbb{Z}_{\geq 0}\}$. Show that B is not context free.