1. Design a data structure that supports the following operations:

- **Insert(x):** insert the key $x$ into the data structure if it is not already there;
- **Delete(x):** delete the key $x$ from the data structure (if it is there);
- **Search(x):** determine if the key $x$ is in the data structure;
- **Select(k):** find the $k$-th smallest key in the data structure.

All these operations should take $O(\lg n)$ time in the worst case, where $n$ is the current number of keys in the data structure.

2. Recall the family of hash functions $\mathcal{H} = \{H_k : k \in K\}$ defined in class: the universe $U$ and key space $K$ consist of all $\ell$-tuples of elements of $\{0, \ldots, m-1\}$; for $x = (x_1, \ldots, x_\ell)$ and $k = (k_1, \ldots, k_\ell)$, we defined $H_k(x) = (x_1k_1 + \cdots + x_\ell k_\ell) \mod m$. We argued in class that if $m$ is prime, then $\mathcal{H}$ is universal. Show that the requirement that $m$ is prime is necessary by showing in particular that when $m = 15$, $\mathcal{H}$ is not universal.

3. Suppose we are using a hash table with $m$ slots, collisions are resolved by chaining, and that we wish to insert a set $S \subset U$ of $n$ distinct keys into the table. Assume that $n \leq m$. To do this, we choose a random hash function $H_k$ out of a universal class $\mathcal{H}$ of hash functions from $U$ into $\{0, \ldots, m-1\}$. For $0 \leq i < m$, let $n_i$ be the number of keys that hash to slot $i$. Let $M = \max\{n_i : 0 \leq i < m\}$.

Show that the expected value $E[M]$ of $M$ is $O(\sqrt{m})$. Use any facts proved in class and in the notes on probability theory.

4. The upper bound in the previous question may not seem very good, but we can prove that it is nearly optimal in the sense that we can construct a universal class of hash functions $\mathcal{H}$ such that $E[M] = \lceil \sqrt{n-1} \rceil$.

The universe is $U = \{0, \ldots, n-1\}$, and we are using a hash table with $n$ slots (so in the usual notation, $m = n = |U|$). Let $1 \leq \ell \leq n$ be an integer. For each $k \subset U$ of cardinality $\ell$, we define a function $H_k : U \to U$ by the following property: $H_k$ maps $k$ onto 0 (i.e., $\forall x \in k : H_k(x) = 0$), and maps $U \setminus k$ injectively into $U \setminus \{0\}$ (i.e., $\forall x \in U \setminus k : (H_k(x) \neq 0) \land (\forall y \in U \setminus k : y \neq x \Rightarrow H_k(x) \neq H_k(y))$). Note that $H_k$ is not uniquely determined by this property, but we can always choose an $H_k$ satisfying this property (verify). Define $\mathcal{H} = \{H_k : k \in K\}$, where $K = \{k \subset U : |k| = \ell\}$.

Notice that for every $H_k \in \mathcal{H}$, when we insert the keys $0, \ldots, n-1$ into the table, $n_0 = \ell$ and $\forall i = 1 \ldots n-1 : n_i \leq 1$; therefore, $E[M] = E[n_0] = \ell$.

(a) Argue that

$$|K| = \binom{n}{\ell}.$$

(b) Argue that for $x, y \in U$ with $x \neq y$

$$|\{k \in \mathcal{H} : H_k(x) = H_k(y)\}| = \binom{n-2}{\ell-2}.$$

(c) From (a) and (b), argue that for all $x, y \in U$ with $x \neq y$,

$$|\{k \in K : (H_k(x) = H_k(y))\}|/|K| = \frac{\ell(\ell-1)}{n(n-1)}.$$

(d) Argue that $\mathcal{H}$ is universal if $\ell \leq \sqrt{n} - 1$. 