A Qualitative Calculus for Three-Dimensional Rotations

Azam Asl
Department of Computer Engineering
NYU Polytechnic
aa2821@cs.nyu.edu

Ernest Davis
Dept. of Computer Science
New York University
New York, NY 10012
davise@cs.nyu.edu

Abstract

We have developed a qualitative calculus for three-dimensional directions and rotations. A direction is characterized in terms of the signs of its components relative to a fixed coordinate system. A rotation is characterized in terms of the signs of the components of the associated 3×3 rotation matrix.

A system has been implemented that can solve the following problems:

1. Given the signs of direction $\hat{v}$ and rotation matrix P, find the possible signs of the image of $\hat{v}$ under P. Moreover, for each possible sign vector of $\hat{v} \cdot P$, generate exact instantiations of $\hat{v}$ and P that yields that result.
2. Given the signs of rotation matrices P and Q, find the possible signs of the composition $P \cdot Q$. Moreover, for each possible sign matrix for the composition, generate exact instantiations of P and Q that yield that result.

We have also proven some related complexity and expressivity results. Determining the satisfiability of a system of equations over signed vectors and rotation matrices is NP-hard. Any two directions are distinguishable by a constraint network of signed vectors and rotations.

Introduction

The field of Qualitative Spatial Reasoning (QSR) develops methods for carrying out geometric computations using qualitative information about spatial properties and relations, rather than numerically precise information (Cohn and Renz 2007). The majority of the QSR literature has addressed reasoning about topological constraints between regions; the best known theory is the RCC-8 system of relations (Randell, Cui, and Cohn 1992). However, other work in the area has addressed other geometric properties such as convexity, relative position, and relative size. A qualitative calculus is a theory that describes how an inference engine can use a constraint network of qualitative geometric relations to draw conclusions that are implicit but not explicit in the network.

The research described in this paper develops the first qualitative calculus for three-dimensional directions and rotations. As every rigid motion is the composition of a rotation and a translation, and as the theory of translation is very simple, developing a qualitative theory of rotation is a significant step toward a system that can reason qualitatively about three-dimensional motion generally. This potentially can have a wide range of applications, from planning to robotics to molecular dynamics. For example if you are working with an articulated robotic arm, and you are considering a partially specified plan in which the angle at each joint is currently partially constrained, then a calculus like ours will allow you to compute the possible orientations of the final segment as the composition of all the rotations at all the intermediate joints. We are not, however, making any strong claims about the immediate applicability of the theory in this paper to any particular practical applications.

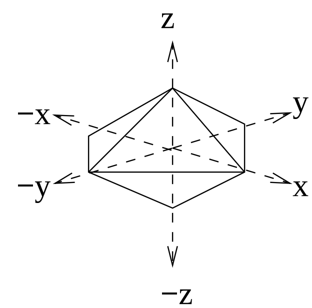

Figure 1: Standard octahedron

Our calculus is based on the well-known sign calculus over the three values +,−, and 0. A standard reference rectangular coordinate system is fixed. A direction $\hat{u}$ is then characterized in terms of the signs of the components of $\hat{u}$ in the x, y, and z direction. There are thus 27 possible combinations of signs; however, since $\langle 0,0,0 \rangle$ is not a direction, there are 26 possible sign vectors for directions.

The sign vectors can usefully be mapped to the vertices, edges, and faces of a standard octahedron, centered at the origin, and with vertices at the points $\hat{x}, -\hat{x}, \hat{y}, -\hat{y}, \hat{z}$ and $-\hat{z}$. A sign vector with two 0 components corresponds to a vertex; for example, $\langle 0, 0, - \rangle$ is the $-\hat{z}$ vertex. A sign vector with one 0 component corresponds to an edge; for example $\langle +, 0, - \rangle$ corresponds to the edge connecting $\hat{x}$ to $-\hat{z}$. A sign vector with no 0 components corresponds to a face; for example, the vector $\langle -, +, + \rangle$ corresponds to the face with vertices $-\hat{x}, \hat{y}, \hat{z}$ (Figure 1).

A three-dimensional rotation Γ is characterized exactly by an orthogonal matrix P, such that for every vector $\vec{v}$, $\Gamma(\vec{v}) = \vec{v} \cdot P$. In our theory, the rotation Γ is characterized qualitatively in terms of the signs of the elements of the matrix P. For example, the rotation matrix

$$M = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}$$

is characterized as the sign matrix

$$\begin{bmatrix} + & + & + \\ + & - & + \\ + & + & - \end{bmatrix}$$

An exact matrix M describes a rotation if it satisfies the following two properties.

A. M is orthogonal. That is $M \cdot M^T = I$. Equivalently, every row and every column has unit length; each row is orthogonal to every other row; and every column is orthogonal to every other column. A consequence is that if two elements in a row/column are zero then the third element must be ± 1, and conversely.

B. The determinant of M is equal to 1. A matrix satisfying (A) is either a rotation or a reflection. Rotations have determinant 1; reflections have determinant -1.

A systematic case analysis shows that there are 336 sign matrices that correspond to possible rotation matrices.

The three rows of a rotation matrix P are respectively the images of the $\hat{x}$, $\hat{y}$ and $\hat{z}$ axes under the rotation $\vec{v} \cdot P$. Therefore the rows of the sign matrix are the sign vectors for $\hat{x} \cdot P$, $\hat{y} \cdot P$ and $\hat{z} \cdot P$.

We have implemented a system called 3DR that can solve the following problems:

1. Given the signs of direction $\hat{v}$ and rotation matrix P, find the possible signs of the image of $\hat{v}$ under P. Moreover, for each possible sign vector of $\hat{v} \cdot P$, generate exact instantiations of $\hat{v}$ and P that yield that result.
2. Given the signs of rotation matrices P and Q, find the possible signs of the composition $P \cdot Q$. Moreover, for each possible sign matrix for the composition, generate exact instantiations of P and Q that yield that result.

The most extensively studied form of QSR in general is the problem of determining the consistency or the consequences of a constraint network. In the context of our theory, such a network would have directions as the nodes, labelled by a sign vector, or by a set of possible sign vectors, and rotations as the edges, labelled by a sign matrix or by a set of sign matrices. Our solution to problem (1) above would allow Waltz propagation to be carried out, to constrain the node labels; our solution to problem (2) would allow arc propagation to be carried out, to constrain the edge labels. That is, the solution to (2) constitutes a transitivity table. We have not implemented either propagation algorithm, but, given the functionalities implemented in 3DR, this is very straightforward; the superroutines to carry out the propagation are entirely standard ((Mackworth and Freuder 1985); (Russell and Norvig 2010) chap. 6).

The transition network over these relations, characterizing continuous change, is easily characterized; the matrix P may instantaneously transition to Q if some of the non-zero signs in Q are changed to 0 in P.

Related work

A number of earlier studies have considered qualitative directional constraints of one form and another, in two dimensional geometry. (Frank 1991) considers relations between points described by the cardinal directions. The STAR calculus of (Renz and Mitra 2004) divides the unit circle of directions into m sectors plus m rays, and considers constraints over a domain of points of the form "The direction from **p** to **q** lies in section k." The $\mathcal{OPRA}$ calculus of (Moratz, Dylla, and Frommberger 2005) uses a similar qualitative division of the unit circle to characterize the relation between directed points. The $\mathcal{LR}$ system of (Scivos and Nebel 2010) uses a system of ternary constraints over points, of the form "Point **r** lies to the left of the directed line going from **p** to **q**."

It should be noted that three-dimensional rotations are intrisically much more complicated than two-dimensional rotations, for three reasons. First, the space of two-dimensional rotations is isomorphic to the space of two-dimensional directions. Second, two-dimensional rotations commute. Third, in two dimensions, both applying a rotation to a direction and composing two directions correspond to the simple operation of adding angles mod 2π. In three dimensions the first two are false, and no simple formulas analogous to the third exist.

Qualitative Rotations

In this section, we describe the implementation of 3DR. We describe first how the symmetries of the geometry can be used to dramatically simplify the case analysis, and then how the reduced case analysis is carried out.

Octahedral Rotations

The *octahedral* rotations map the standard octahedron into itself; equivalently, they map $\hat{x}$, $\hat{y}$, $\hat{z}$ into some combination of $\pm\hat{x}$, $\pm\hat{y}$, $\pm\hat{z}$. There are 24 such rotations. Proof: You can choose the image $\Gamma(\hat{x})$ to be any of the 6 vertices of the octahedron; then you can choose $\Gamma(\hat{y})$ to be any of the 4 orthogonal vertices; then $\Gamma(\hat{z})$ is fixed.

The matrix for an octahedral rotation has one non-zero element, either 1 or -1, in each row and column. For example, the rotation Γ such that $\Gamma(\hat{x}) = -\hat{z}, \Gamma(\hat{y}) = \hat{y}, \Gamma(\hat{z}) = \hat{x}$ corresponds to the matrix

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

There are 24 such matrices with determinant 1.

If $\hat{u}$ and $\hat{v}$ are directions on the same piece of the octahedron, and Γ is an octahedral rotation then clearly $\Gamma(\hat{u})$ and $\Gamma(\hat{v})$ are still in the same piece of the octahedron. Therefore, if A and B are two exact rotational matrices with the same signs, and P is an octahedral rotation matrix, then $A \cdot P$

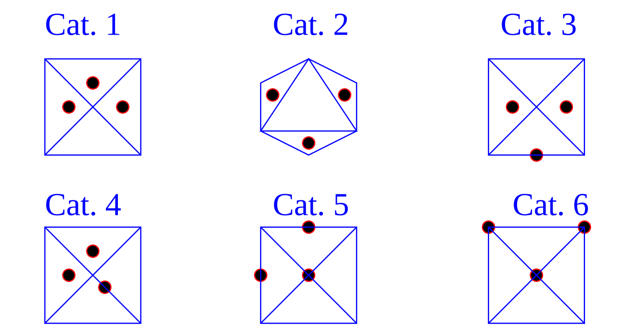

Figure 2: Categories of rotations. The lines are the standard octahedron; the red dots are the images of $\hat{x}, \hat{y}, \hat{z}$

and $B \cdot P$ have the same sign. Therefore, we can define an equivalence relation over the sign rotation matrices: Q is equivalent to R if $Q = R \cdot P$ for some octahedral rotation P. Every equivalence class of sign rotation has 24 elements; since there are 336 sign rotation matrices, there are 336/24=14 equivalence classes. We identify one representative $Q_1 \ldots Q_{14}$ from each of these equivalence classes, and then we can write any sign rotation matrix in the form $Q \cdot P$ for some representative Q and some octahedral rotation P. Since the octahedral rotations are easy to deal with, this greatly simplifies the analyis of the sign rotation matrices.

Since the octahedral rotations preserve the geometry of the octahedron, we can be sure that two sign rotations A and B are from different equivalence classes if the two triples $\langle \hat{x} \cdot A, \hat{y} \cdot A, \hat{z} \cdot A \rangle$ and $\langle \hat{x} \cdot B, \hat{y} \cdot B, \hat{z} \cdot B \rangle$ are geometrically different, in terms of features of the octahedron. Note that these are the rows of A and B respectively.

Below is a list of 14 representatives, with geometric features that guarantee that they are distinct. These fall into 6 categories (figure 2).

Category 1 includes three representatives. In these. $\hat{x} \cdot R$, $\hat{y} \cdot R$, and $\hat{z} \cdot R$ are all in faces that share the vertex $\langle 1, 0, 0 \rangle$.

```
       + + +          + + +          + + +
R1 = + - +    R2 = + - +    R3 = + - -
       + + -          + - -          + + -
```

Note that the face corresponding to $\hat{x} \cdot R1$ shares an edge with both $\hat{y} \cdot R1$ and with $\hat{z} \cdot R1$, but that $\hat{y} \cdot R1$ and $\hat{z} \cdot R1$ do not share an edge; that $\hat{y} \cdot R2$ shares an edge with both $\hat{x} \cdot R2$ and with $\hat{z} \cdot R2$; $\hat{z} \cdot R3$ shares an edge with both $\hat{x} \cdot R3$ and with $\hat{y} \cdot R3$. (Two faces share an edge if the sign vectors are equal in two places; they share a vertex if the sign vectors are equal in one place; they are antipodal if the signs are opposite throughout.)

Category 2 includes a single representative. In this $\hat{x} \cdot R$, $\hat{y} \cdot R$, and $\hat{z} \cdot R$ are all inside faces. Each of the faces connects to each of the other at a vertex.

```
       + + +
R4 = + - -
       - + -
```

Category 3 includes 3 representatives. In these, two of the row vectors lie in faces F1, F2 and one lies in an edge E. Faces F1, F2 have a common vertex V; edge E connects a vertex of F1 not equal to V with a vertex of F2 not equal to V. For instance in R5 below, F1 (the second row) is the face with vertices $-\hat{x}$, $\hat{y}, \hat{z}$; F2 (the third row) is the face with vertices $\hat{x}$, $-\hat{y}, \hat{z}$; these meet at vertex V=$\hat{z}$. Edge E (the first row) connects $\hat{x}$ with $\hat{y}$.

```
       + + 0          + + +          + + +
R5 = - + +    R6 = + 0 -    R7 = - - +
       + - +          - + -          + - 0
```

Category 4 includes 3 representatives. In these, two of the row vectors lie in a face and one lies in an edge. The two faces have a common edge; one vertex of the edge meets one of the shared vertices of the faces.

```
       + + +          + + +           + + 0
R8 = + + -    R9 = - 0 +    R10 = - + -
       - + 0          + - +           - + +
```

Category 5 includes 3 representatives. In these, one vector is mapped to a vertex and the other two are mapped to edges. The representatives are rotations around the coordinate axes.

```
        + 0 0           + 0 -           + - 0
R11 = 0 + -    R12 = 0 + 0    R13 = + + 0
        0 + +           + 0 +           0 0 +
```

Category 6 is the class of octahedral rotation. The single representative is the identity. All the vectors are mapped to vertices.

```
        + 0 0
R14 = 0 + 0
        0 0 +
```

For any exact matrix or sign matrix M and octahedral rotation R, the product $M \cdot R$ is simply a permutation of the columns of M, possibly with one or more change of sign, and the product $R \cdot M$ is a permutation of the rows of M, possibly with sign change . Likewise, the product $\hat{u} \cdot R$ is a permutation of the elements of $\hat{u}$, possibly with sign changes. In particular, all three products are always uniquely valued, even when $\vec{u}$ is a sign vector or M is a sign matrix.

Using Octahedral Symmetry

Having identified the octahedral rotations and a representative from each equivalence class, we can use this analysis to simplify the computation of operations on sign vectors and sign rotation matrices. In particular we define the representative sign vectors to be the vectors $\langle +, 0, 0 \rangle$, $\langle +, +, 0 \rangle$ and $\langle +, +, + \rangle$. We can then use the octahedral rotations to map any problem involving sign vectors and matrices to an equivalent problem involving the representative sign vectors and sign matrices. Thus the number of problems of the form $\hat{u} \cdot P$ is reduced from $26 \cdot 336$ to $3 \cdot 14$, a simplification by a factor of 208, and the number of problems of the form $P \cdot Q$ is reduced from $336 \cdot 336$ to $14 \cdot 14$, a simplification by a factor of 576.

Given the problem, "Compute the possible signs of $\hat{v} \cdot P$," where $\hat{v}$ is a sign vector and P is a sign matrix, we proceed as follows.

1. Let $\hat{u}$ be the representative sign vector with the same number of 0's as $\hat{v}$. Find an octahedral rotation R such that $\hat{v} = \hat{u} \cdot R$.

2. Compute the product $S = R \cdot P$. This is a rotation matrix.
3. Factor the matrix S as the product of a a rotation representative Q and a octahedral rotation R': $S = Q \cdot R'$.
4. Compute the possible values of $\hat{w} = \hat{u} \cdot Q$. Note that this is the product of a representative sign vector with a representative sign matrix.
5. Return $\hat{w} \cdot R'$. Note that $\hat{w} \cdot R' = \hat{u} \cdot Q \cdot R' = \hat{u} \cdot R \cdot P = \hat{v} \cdot P$.

Given the problem "Compute the possible signs of $P \cdot Q$" where P and Q are sign matrices, we proceed as follows:

1. Factor $P = P_1 \cdot R_1$ where P_1 is a representative matrix and R_1 is an octahedral rotation.
2. Compute $Q_1 = R_1 \cdot Q$.
3. Factor $Q_1 = Q_2 \cdot R_2$ where Q_2 is a representative sign matrix and R_2 is an octahedral rotation.
4. Compute $W = P_1 \cdot Q_2$. Note that this is the product of two representative matrices.
5. Return $W \cdot R_2$.

We have thus reduced these two problems to the following subproblems:

A. Multiplying a sign vector or a sign matrix by an octahedral rotation. As discussed above, this is simply a permutation plus changes of sign.

B. Factor a sign vector $\vec{v} = \hat{u} \cdot R$ where $\hat{u}$ is a representative sign vector and R is an octahedral rotation. This is easily precomputed; there are 26 cases.

C. Factor a sign matrix $P = P' \cdot R$ where P' is a representative matrix and R is an octahedral rotation. This can be precomputed by computing $Q' \cdot R$ for all pairs of a representative matrix P and an octahedral rotation R; there are 336 cases.

D. Multiply a representative sign vector by a representative sign matrix or multiply two representative sign matrices. These are precomputed; there are $3 \cdot 14 = 42$ and $14 \cdot 14 = 196$ cases respectively. However, the analysis here deserves its own section.

Multiplying representatives

We have thus reduced our two problems to the problem of multiplying a representative sign vector by a representative sign matrix and the problem of multiplying two representative matrices.

Multiplying a vector by a matrix One obvious approach to these problems is to use the standard methods for computing products, applying the standard sign calculus for combining signs. For example, let

$$\hat{v} = [+,+,0] \text{ and let } R4 = \begin{bmatrix} + & + & + \\ + & - & - \\ - & + & - \end{bmatrix}$$

Then multiplying in the usual way by taking the dot product of $\hat{v}$ with each column of $R4$, and using the sign arithmetic we get

$$\begin{aligned} &\hat{v} \cdot R4 = \\ &[(+ \otimes +) \oplus (+ \otimes +) \oplus (0 \otimes -); \\ &\ (+ \otimes +) \oplus (+ \otimes -) \oplus (0 \otimes -); \\ &\ (+ \otimes +) \oplus (+ \otimes -) \oplus (0 \otimes -)] = \\ &[+ \oplus + \oplus 0; + \oplus - \oplus 0; - \oplus + \oplus 0] = [+, I, I]. \end{aligned}$$

In the above formula, we have used $\oplus$ and $\otimes$ for the operators on signs and I as the symbol for "indefinite".

This calculation would suggest that the product may take on any of 9 sign values, all combinations of the three signs for each of the I values. However, this process loses information; all of these products would be attainable if $R4$ were an *arbitrary* matrix with these signs, but not if $R4$ is restricted to be a *rotation* matrix. In fact, of the 9 possible combinations, only 5 are actually possible. The calculation, however, obviously gives a *necessary* condition; no combination of signs that lies outside $[+, I, I]$ is possible.

The following theorem illustrates that the above product generates values that are in fact impossible:

Theorem 1. *If Sign*$(\hat{v}) = [+,+,0]$ *and* M *is a rotation matrix such that Sign*$(M) = R4$*, then Sign*$(\hat{v} \cdot M)$ *is not* $[+,0,0], [+,0,+], [+,-,0]$ *or* $[+,-,+]$.

Proof: Let $\vec{v} = [v_1, v_2, 0]$ and let

$$M = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & -y_2 & -z_2 \\ -x_3 & y_3 & -z_3 \end{bmatrix}$$

where all the variables are positive. Thus $\hat{u} = \hat{v} \cdot M = [v_1 x_1 + v_2 x_2, v_1 y_1 - v_2 y_2, v_1 z_1 - v_2 z_2]$

Since M is orthogonal, the dot product of the third row with the first and second is 0 so

$$-x_2 x_3 - y_2 y_3 + z_2 z_3 = 0 \rightarrow z_2 z_3 > y_2 y_3$$

$$-x_1 x_3 + y_1 y_3 - z_1 z_3 = 0 \rightarrow y_3 y_1 > z_3 z_1$$

Combining these we get $y_1 y_3 z_2 z_3 > y_2 y_3 z_1 z_3$ so $z_2/z_1 > y_2/y_1$

Now if $u_2 \leq 0$ we have $v_1 y_1 \leq v_2 y_2$, so $v_1/v_2 \leq y_2/y_1 < z_2/z_1$, so $v_1 z_1 < v_2 z_2$ so $u_3 < 0$. Therefore none of the signed vectors $[+,0,0], [+,0,+], [+,-,0]$, and $[+,-,+]$ can be the value of Sign$(\hat{u})$. ▮

Establishing that a particular sign vector $\hat{u}$ is a possible product of $\hat{v} \cdot M$ is done by finding exact instantiations of the $\hat{u}$, $\hat{v}$ and M with the desired properties: The three instantiations must have the specified signs; $\hat{u}$ must be equal to $\hat{v} \cdot M$; and M must be an orthogonal matrix. We have implemented a search method that combines three techniques:

- If a variable has a non-zero sign, assign it a random exact value with that sign.
- Value propagation: If there is an equation where all but one variable has been assigned an exact value, then solve for the value of the remaining variable We use use equations of three kinds:
  - $\hat{u} \cdot M[:, j] = \hat{v}[j]$.
  - The dot product of any two rows/columns of M is zero.
  - Each row/column of M is plus or minus the cross product of the other two rows.

- Perturbation. Let $p \cdot A = q$ be a sign equation with $q[3] = 0$, and suppose that you have found a solution $\hat{u}, M$ where Sign$(\hat{u}) = p$, Sign$(M) = A$, Sign$(\hat{u} \cdot M) = q$. Let q' be the same as q except that $q'[3] \neq 0$. Then unless $\hat{u}[i] \cdot M[i, 3] = 0$ for all $i = 1, 2, 3$, it is possible to perturb $\hat{u}$ and M to a solution of the equation $p \cdot A = q'$. Similarly, one can perturb away from 0 in other positions in q and in p and A.

As a final step, $\hat{u}$ and M can be normalized so that $\hat{u}$ and each row and column of M have magnitude 1.

We have succeeded in finding, for every "equation" of the form $\hat{u} = \hat{v} \cdot R$, either an instantiation proving that the equation is satisfiable or an algebraic proof analogous to the one above proving that the equation is unsatisfiable. Over the space of 3 representative vectors $\hat{v}$ and 14 representative matrices M, there are a total of 154 such valid equations. See (Asl 2011) for details.

Multiplying two matrices

The problem of finding all possible sign matrices that can be the values of a product $P \cdot Q$ of two particular representative matrices is much harder.

Three types of constraints are easily found:

1. The result $P \cdot Q$ must be one of the 336 sign rotation matrices.
2. If we decompose the matrix P into rows then the product $P \cdot Q$ decomposes into the individual products of each row of P with Q.

$$\text{If } P = \begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \hat{v}_3 \end{bmatrix} \text{ then } P \cdot Q = \begin{bmatrix} \hat{v}_1 \cdot Q \\ \hat{v}_2 \cdot Q \\ \hat{v}_3 \cdot Q \end{bmatrix}$$

   We can then use the methods described in the previous section to limit the possible values of $\hat{v}_i \cdot Q$. (The rows $\hat{v}_i$ are not in general representative vectors, but we can use the octahedral rotations to transform the problem into one with representative vectors, as described earlier.)
3. We can write $P \cdot Q = (Q^T \cdot P^T)^T$, and then divide Q^T into rows and proceed as in part 2 above. (Again P^T may not be a representative matrix, but again that does not matter.)

All three of these give necessary conditions on the possible signs of the product, and, in general, they give different constraints, so applying them all limits the possible values of the signs. However, they do not together give sufficient conditions; a result may satisfy all three conditions and yet not be a possible product.

The algebraic analysis of this problem becomes quite formidable, and we have not found algebraic arguments analogous to those in the previous section that will allow us to rule out additional impossible values. (It might be worthwhile using an algebraic theorem prover; we have not attempted this.) Rather, we have implemented a randomized procedure for searching for instantiations; details are given in (Asl 2011). If the procedure does not find an instantiation for a sign equation after a specified number of attempts, we presume that no such instantiation exists.

These cases, where a possible value cannot be excluded by the above constraints and has not been instantiated by our search procedure, thus represent a gap in our analysis; in each individual case, it seems likely that there is actually no solution, though we would certainly hesitate to claim that there are none such in the whole collection. Out of 2782 equations that satisfy the above constraints, we have found instantiations for 2604; there are thus 178 unresolved cases.

All the solutions we have found, both of vector times matrix and of matrix times matrix, have been saved in a database, together with a sample instantiation.

Code

The code used to find the solutions, a database containing the solutions of the representative problems, and a program with a user interface to find the solution to a specific problem, are available online at (URL suppressed for blind review). The code is 2000 lines of Java in total.

Complexity and expressivity results

We have shown a number of complexity and expressivity results for existential languages over this theory. Proofs are given in (Asl and Davis 2012). We begin with a complexity hardness result.

Theorem 2. *Consider a system of equations over a collection of variables $\hat{v}_1 \ldots \hat{v}_k$ ranging over three-dimensional directions and a collection $M_1 \ldots M_q$ ranging over three-dimensional rotations, of the following forms:*

- $\hat{v}_a \cdot M_b = \hat{v}_c$;
- *Sign*$(\hat{v}) = s$ *where s is a constant sign vector.*
- *Sign*$(M) = m$ *where m is a constant sign matrix.*

Then the problem of determining whether the system of equations is consistent is NP-hard.

The proof uses a straightforward reduction from oriented matroids (Wolter and Lee 2010). We have no reason to suppose that this problem is in NP. Note that the statement of the theorem allows the sharing of matrix variables between equations; this is a more expressive representation than the constraint network described earlier in which a matrix is associated with a single arc between nodes. We have not found a significant complexity result for the latter, more restricted language.

We next turn to two expressivity results for a much more restricted language. We begin by defining the concepts of two values being distinguishable by a constraint network, and of a value being uniquely identifiable.

Definition 1. *Consider a language $\mathcal{L}$ containing a fixed collection of unary and binary relations over a domain $\mathcal{D}$. A* constraint network *N over $\mathcal{L}$ is a directed graph whose vertices correspond to variables over $\mathcal{D}$ and are labelled with unary relations in $\mathcal{L}$ and whose arcs are labelled with binary relations in $\mathcal{L}$. An* instantiation *of N is an assignment of values in $\mathcal{D}$ to the nodes of N such that all the labels are satisfied.*

Definition 2. *Let $\mathcal{L}$ be a language over $\mathcal{D}$. A value $a \in \mathcal{D}$ is* distinguishable *from $b \in \mathcal{D}$ by constraint networks over $\mathcal{L}$*

iff there exists a network N over $\mathcal{L}$ and a node V in N such that

- *there exists an instantiation of N in which the value of V is a; and*
- *there does not exist an instantiation of N in which the value of V is b.*

Definition 3. *A value $a \in \mathcal{D}$ is* uniquely identifiable *by constraint networks over $\mathcal{L}$ if there exists a network N and node V in N such that all instantiations of N assign a to V.*

We now define a language $\mathcal{L}_1$ with a very limited vocabulary of qualitative rotation relations. Let $\mathcal{D}$ be the space of three-dimensional directions. Let $\mathcal{Q}$ be the space of rotations around one of the coordinate axes (these have sign matrices with four 0's; they are in category 5, discussed above). Let $\mathcal{L}_1$ be the language with the following relations:

- Unary relations: For any sign vector s, the relation $\text{Sign}(\hat{v}) = s$.
- Binary relations: For any sign matrix s corresponding to a rotation in $\mathcal{Q}$, the relation $P(\hat{u}, \hat{v})$ iff there exists a rotation $Q \in \mathcal{Q}$ such that $\text{Sign}(Q) = s$ and $\hat{u} \cdot Q = \hat{v}$.

Despite the very limited class of rotations considered (only rotations around the coordinate axes, not rotations around any other axes), the following theorems hold:

Theorem 3. *Any direction is distinguishable from any other direction by constraint networks over $\mathcal{L}$.*

Theorem 4. *The set of directions uniquely identifiable by constraint networks over $\mathcal{L}$ is dense in the unit sphere.*

Essentially, rotations in $\mathcal{Q}$ allow one to construct circles on the unit sphere centered at the coordinate axes, and the proof consists in showing that there are uniquely identifiable patterns among these circles.

Theorems 3 and 4 remain true if domain $\mathcal{D}$ is limited to the non-negative octant (that is, the set of directions in which $\hat{x}$, $\hat{y}$ and $\hat{z}$ components are all non-negative). However, over that restricted range, despite the strong expressivity result, the theory is polynomial-time decidable. Our final theorem is this tractability result.

Theorem 5. *Let $\mathcal{D}^+$ be the non-negative octant, and let $\mathcal{L}_1^+$ be $\mathcal{L}_1$ restricted to $\mathcal{D}^+$. The problem of determining whether a constraint network over $\mathcal{L}_1^+$ is consistent (has an instantiation) is decidable in polynomial time.*

It turns out, in fact, that the problem in theorem 5 is reducible to solving a system of linear inequalities, where the variables in the inequalities are the squares of the components of the vectors.

Certainly it would be of interest to get stronger, or additional results of this kind. Of particular interest would be to consider the consequence of considering, the space of rotations with no 0's, over the strictly positive octant, rather than the space of rotations with four 0's used above. In this modified problem, the analogue of theorem 4 fails in the strongest possible way; there are *no* uniquely identifiable directions. However we do not know anything about the analogue of theorem 3.

Future Work

The immediate problems to be addressed would be to close the gaps in our analysis of signed matrix composition, and to extend the complexity/expressivity results discussed in the previous section.

It would also be helpful to be able to generate *random* solutions to a specified sign equation. It is not particularly important to achieve any particular distribution (e.g. uniform), but it would be desirable to have a program that at least achieves reasonable coverage; that is, for any given sign equation Q and for any instantiation I satisfying Q, in some reasonable number of trials the program will output a solution to Q fairly close to I. The use of perturbation in the current technique for finding instantiations has the consequence that, in some cases, instantiations far from 0 may not be found.

More interestingly, the theory can be extended to other tilings of the unit sphere beyond the octahedral, though the sign calculus would no longer be applicable. Indeed a very similar theory can be developed for any of the regular (Platonic) polyhedra. Consider, for example, a fixed icosahedron centered at the origin, with individual names assigned to each vertex, edge, and face. A direction can be qualitatively characterized by identifying the vertex, edge, or face it passes through. A rotation can be characterized as follows. Pick a particular face of the icosahedron, and let $\hat{p}$, $\hat{q}$, $\hat{r}$ be the vertices of that face. Then characterize a rotation Γ in terms of the qualitative characterization of $\Gamma(\hat{p}), \Gamma(\hat{q})$ and $\Gamma(\hat{r})$.

The resultant theory is then very similar to the one we have developed here. The symmetry group of the icosahedron has 60 elements. There are still three representative qualitative directions, corresponding to vertex, edge, and face. By our count, there are 23 representative rotations in 9 categories. The geometric calculations involved are, of course, much harder than in our analysis.

Qualitative representations based on the Platonic solids can only achieve a fixed level of granularity, as there are only five Platonic solids. One can achieve arbitrary levels of granularity by using a finer tiling of the unit sphere, at the cost of losing some of the symmetries. Choose a reference tiling, and three reference directions $\hat{p}, \hat{q}, \hat{r}$; characterize a direction in terms of the element of the tiling that it lies in; and characterize a rotation in terms of the characterizations of the images of $\hat{p}, \hat{q}, \hat{r}$.

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