Universal Configurations in Light-Flipping Games

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1 "Lights Out" and "Orbix"

The following is a popular hand-held electronic game by Tiger Electronics, called "Lights Out". This game is played on a 5×5 grid of buttons which also have lights in them. By pressing a button, its light and those of the (non-diagonally) adjacent buttons will change (switch ON if it was OFF, and vice versa). Given some initial pattern of lights, one has to switch them all OFF by pressing several buttons. Obviously, the game can be played on other boards (indeed, Tiger produced a 6×6 version, and even a $3 \times 3 \times 3$ cube version), and naturally generalizes to any graph G. In addition, there are many on-line implementations and other interesting documentation about the game (see [4] and the numerous links therein, or simply search the web for "lights out tiger").

On a more scientific front, "Lights Out" and the questions derived from it (e.g., which configurations can be turned off for which graphs, how many buttons does one have to press, what is the smallest number of lights that can be left ON, etc.) have generated a surprising amount of research (see [5] and the references therein). We shall point out only one somewhat surprising fact, first discovered by Sutner [3], and later simplified by [1, 2]. Namely, while many initial configurations cannot be completely turned off for many graphs G, it turns out that the "all-ON" configuration *can always be turned off, for any n-vertex graph G*. We will call such a configuration *universal*. We notice that another (trivial) universal configuration is the "all-OFF" configuration. By looking at the complete graph G (which has only two opposite configurations for any initial configuration of lights), we see that all-ON and all-OFF are the only universal configurations for the "Lights Out" game.

"Orbix" is another very similar sounding electronic game produced by Meffert's [6]. Now the basic game is played on the icosahedron rather than a grid. But the rules are the same except for one major difference: pressing a button only changes the state of the neighboring buttons, but not the state of the actual button pressed. Again, "Orbix" obviously generalizes to any graph G. Now, however, the only universal configuration (the one that can be completely turned off for any G) is the trivial all-OFF configuration (which can be seen by looking at the empty graph, where any initial configuration can not be changed in "Orbix").

2 Common Generalization

We now consider the following common generalization of the above two games, which we call the *light-flipping* game. As before, we are given some undirected n-vertex graph G, each of whose nodes has an indicator light (which can be ON or OFF) and a button. However, now the buttons are of two possible types: *exclusive* and *inclusive*. Pressing any exclusive button will flip the state of all of the neighboring buttons in G (from ON to OFF and vice versa), but leaves unchanged the state of the button pressed (ala "Orbix"). On the other hand, pressing any inclusive button will also flip the state of this button as well (ala "Lights Out"). Let b be the vector of button types, i.e. $b_v = 0$ if button v is exclusive, and $b_v = 1$ for an inclusive v. Given some initial configuration c of lights, the objective of the light-flipping game is to turn all the lights OFF by pressing several buttons.

Definition 2.1. *A configuration c is called* universal *for a button pattern b, if c can be turned off for any graph G.*

In this paper we determine all universal configurations for arbitrary light-flipping games (given by b). Given a configuration of lights c, we will write $c_v = 1$ if v's light is ON, and $c_v = 0$ if it is OFF. Then, our main result is given by the following

Theorem 2.1. *The only universal configurations c for a given button pattern b are $c = \vec{0}$ (the trivial all-OFF configuration) and $c = b$. In particular, for any graph G one can turn off all the lights when $c = b$.*

Notice that our result generalizes the result of Sutner [3] for the "Lights Out" game, stating that $c = \vec{1}$ (all-ON) is a universal configuration for this game. Also, any light-flipping game other than "Orbix" has a (unique) *non-trivial* universal configuration $c = b$. In

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other words, a configuration that is ON on inclusive buttons, and OFF on exclusive buttons can always be turned off!

3 Proof of Main Theorem

Take any button pattern b. Let us start with determining which initial configurations could be universal. Consider a graph G whose (only) edges form a clique K on all the inclusive buttons. In particular, all the exclusive buttons are the isolated vertices of G. Assume that an initial configuration c can be turned off for this G (and b). Then all the exclusive (isolated) buttons should be OFF in c, since there is no other way to turn them off. On the other hand, since pressing any inclusive button simultaneosly flips the state of *all* the inclusive buttons (since G forms a clique on these buttons), and since the all-OFF state has to be reached, there are only two possibilities for c on K: either all the inclusive buttons should be OFF (this gives $c = \vec{0}$), or all should be ON (this gives $c = b$).

Before showing the converse, we rewrite our problem using some linear algebra. First, it never makes sense to press a button twice (which is the same as not pressing the button at all), and the order of the buttons is not important as well. Thus, turning off an initial configuration c is equivalent to finding a *subset* S of buttons to press. Take any such candidate S and let x be the characteristic vector of S: $x_w = 1$ if $w \in S$ and $x_w = 0$ otherwise. Then the final status of a light at v after pressing buttons in S is simply

$$c_v + \sum_{w \in N(v)} x_w + b_v x_v \tag{3.1}$$

where the addition modulo 2, and $N(v) = \{w \mid (v,w) \in E(G)\}$ is the set of v's neighbors in G. Indeed, c_v was v's initial state, the sum in the middle is the contribution of v's neighbors in G, and $b_v x_v$ is the contribution from possibly pressing the button v itself. But Equation (3.1) is just an affine linear transformation over $GF(2)$! Namely, if we let $A = A(G,b)$ be the $n \times n$ adjacency matrix of G with the vector b on the diagonal (i.e., for $v \neq w$, $a_{v,w} = 1$ iff $(v,w) \in E(G)$, and $a_{v,v} = b_v$), the final light configuration of G will be $(Ax + c)$, where all the operations are over $GF(2)$. Hence, a set S turning all the lights off (i.e., making $Ax + c = \vec{0}$) exists iff *the linear system* $Ax = c$ *is solvable over* $GF(2)$. Namely,

LEMMA 3.1. *Let b be a button pattern, G be a graph and $A = A(G,b)$. An initial configuration c can be turned off if and only if the linear system $Ax = c$ is solvable over $GF(2)$.*

As a side remark, the above lemma gives an efficient procedure to turn all the lights off, when possible. But let us get back to the converse of our theorem. Since $\vec{0}$ is a trivial universal configuration, it remains to show that one can always turn off the configuration $c = b$. Notice that by varying the graph G, the matrix $A = A(G,b)$ ranges through all the symmetric matrices over $GF(2)$ whose whose diagonal (denoted $\mathrm{diag}(A)$) is equal to b. Thus, to show that the configuration $c = b$ is universal it is (necessary and) sufficient to show the following lemma of independent interest:

LEMMA 3.2. *Let $A = (a_{ij})$ be an $n \times n$ symmetric zero-one matrix, and let $b = \mathrm{diag}(A)$. Then the linear system $Ax = b$ is solvable over $GF(2)$.*

Proof. Assume on the contrary that the system is not solvable. This means that there are linearly dependent rows $i_1, \ldots, i_k$ of A which yield a contradiction, i.e. $b_{i_1} + \ldots + b_{i_k} \neq 0$. Since $b = \mathrm{diag}(A)$, we get that

$$a_{i_1,i_1} + \ldots + a_{i_k,i_k} = 1 \tag{3.2}$$

Let A' be a square sub-matrix of A generated by rows $i_1, \ldots, i_k$ and columns $i_1, \ldots, i_k$. Let us denote by s the sum of all the elements of A', and compute s in two different ways. First, since the rows $i_1, \ldots, i_k$ of A are linearly dependent, each column of A' sums to 0, making $s = 0$.

On the other hand, since A is symmetric, then so is A'. But the sum of all the entries of a symmetric matrix over $GF(2)$ equals to the sum of its diagonal entries! Indeed, all the off-diagonal elements are summed twice canceling each other over $GF(2)$. Hence, $s = a_{i_1,i_1} + \ldots + a_{i_k,i_k}$, which is equal to 1 by Equation (3.2). Hence, we got both $s = 0$ and $s = 1$, a contradiction.

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